21Th APRIL 2006

INDEX

S.No	TOPICS	Page No.
1	CHAPTER NO. I-A Coordinate System	01
2	CHAPTER NO. I Survey of Elementry Principle	07
3	CHAPTER NO. II CONSTRAINTS	14
4	CHAPTER NO. III VARIATIONAL PRINCIPLES AND LANGRANGES EQUATION	41
5	CHAPTER NO. IV HAMILTON'S EQUATION OF MOTION	58
6	CHAPTER NO. V MOTION IN A CENTRAL FORCE FIELD	75
7	CHAPTER NO. VI Collisions of Particles	98
8	CHAPTER NO. VII POISSON BRACKETS	113.

12° Classical-M ①

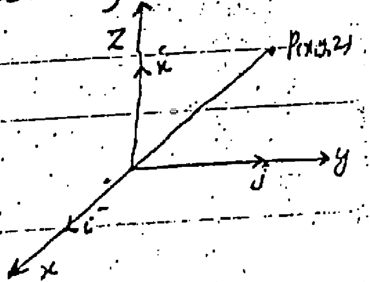
27.01.06

CHAP NO. 1

SURVEY OF THE ELEMENTRY PRINCIPLE

PRE REQUISITE Coordinate SYSTEM

① Cartesian Coordinate System :- (x, y, z)



The cartesian coordinate system consist of three mutually perpendicular lines. They are represented by (x, y, z). The $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors in the increasing direction of x, y and z respectively. This System is a fixed Coordinate system. The direction of $\hat{i}$, $\hat{j}$ and $\hat{k}$ remain fixed.

we know That $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

The unit vector will be $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

The velocity at Point P is given as

$$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\Rightarrow \vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

Similarly the acceleration will be the double derivative of $\vec{r}$ with respect to t

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

Thus The position, velocity and acceleration as a whole as given as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

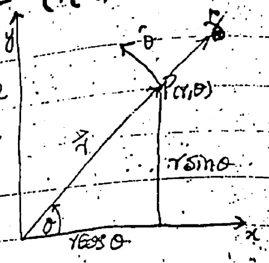
$$\boxed{\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}}$$

$$\boxed{\vec{a} = \ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}}$$

* * *

28.01.06

② Plane Polar Coordinates :- (r,θ)



Consider a Point $P(r,\theta)$ in a plane. In a plane Polar Coordinate its Transformation equations will be

$x = r\cos\theta,\quad y = r\sin\theta$

where r varies from $0 \to \infty$
and θ varies from $0 \to \pi$

$\vec r = x\hat i + y\hat j$

$\vec r = r\cos\theta\,\hat i + r\sin\theta\,\hat j$

$\vec r = r(\cos\theta\,\hat i + \sin\theta\,\hat j)$

we know That The unit vector is equal to The vector divided by its megnitude

$\hat r = \vec r/|r|$

$\Rightarrow \vec r = r\hat r$ putting values of $\vec r$

$r(\cos\theta\,\hat i + \sin\theta\,\hat j) = r\,\hat r$

$\Rightarrow \hat r = \cos\theta\,\hat i + \sin\theta\,\hat j$ —①

Here in The figure we know That $\hat\theta$ is at The right angle (θ is 90°) to $\hat r$ Then we have

$\hat\theta = \cos(\theta + \pi/2)\,\hat i + \sin(\theta + \pi/2)\,\hat j$

$= (\cos\theta\cos\pi/2 - \sin\theta\sin\pi/2)\,\hat i + (\sin\theta\cos\pi/2 + \cos\theta\sin\pi/2))$

$= (0 - \sin\theta)\,\hat i + \cos\theta\,\hat j$

$\hat\theta = -\sin\theta\,\hat i + \cos\theta\,\hat j$ —②

Now we find The unit vectors by other method

$\hat r = \frac{\vec r}{|r|} = \frac{d\bar r/dr}{|d\bar r/dr|}$ —Ⓐ As $\vec r = \frac{d\bar r}{dr}$

$\frac{d\bar r}{dr} = (\cos\theta\,\hat i + \sin\theta\,\hat j)\frac{dr}{dr}$ $\quad |r| = \left|\frac{d\bar r}{dr}\right|$

$\Rightarrow \frac{d\bar r}{dr} = \cos\theta\,\hat i + \sin\theta\,\hat j$

& $\left|\frac{d\bar r}{dr}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ putt in eq Ⓐ

$\hat r = \frac{\cos\theta\,\hat i + \sin\theta\,\hat j}{1}$

$\Rightarrow \hat r = \cos\theta\,\hat i + \sin\theta\,\hat j$

29.01.06

Then The $\hat\theta$ will be $\hat\theta = \frac{d\vec r/d\theta}{|d\vec r/d\theta|}$ —Ⓑ

$\vec r = r(\cos\theta\,\hat i + \sin\theta\,\hat j)$

$\frac{d\vec r}{d\theta} = r(-\sin\theta\,\hat i + \cos\theta\,\hat j)$

& $\left|\frac{d\vec r}{d\theta}\right| = \sqrt{r^2(\sin^2\theta + \cos^2\theta)} = \sqrt{r^2(1)} = r$

putting both This values in eq Ⓑ we get

$\hat\theta = \frac{r(-\sin\theta\,\hat i + \cos\theta\,\hat j)}{r} = -\sin\theta\,\hat i + \cos\theta\,\hat j$

$\Rightarrow \hat\theta = -\sin\theta\,\hat i + \cos\theta\,\hat j$

Now we are going to find out The velocity and acceleration which are our main objectives

As $\vec r = r\hat r$

The velocity will be $v = \dot{\vec r} = \frac{d\vec r}{dt} = \frac{d}{dt}(r\hat r)$

$\dot{\vec r} = \dot r\hat r + r\frac{d}{dt}\hat r$

$= \dot r\hat r + r\frac{d\hat r}{d\theta}\cdot\frac{d\theta}{dt}$ $\quad$ As $\left(\frac{d\theta}{dt} = \dot\theta\right)$ $\left(\frac{d\hat r}{d\theta} = \hat\theta\right)$

$v = \dot{\vec r} = \dot r\hat r + r\hat\theta\dot\theta$

$\Rightarrow \boxed{v = \dot{\vec r} = V_r\hat r + V_\theta\hat\theta}$

Where $V_r = \dot r$ which is Radial velocity

& $V_\theta = r\dot\theta = r\omega$ is Angular velocity $\left(\omega = \dot\theta = \frac{d\theta}{dt}\right)$

Now we are going to find acceleration

$\vec a = \frac{d}{dt}\vec v = \frac{d}{dt}(\dot r\hat r + r\dot\theta\hat\theta)$

$\vec a = \ddot r\hat r + \dot r\frac{d\hat r}{dt} + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta + r\dot\theta\frac{d\hat\theta}{dt}$

$= \ddot r\hat r + \dot r\frac{d\hat r}{d\theta}\cdot\frac{d\theta}{dt} + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta + r\dot\theta\frac{d\hat\theta}{d\theta}\cdot\frac{d\theta}{dt}$

$= \ddot r\hat r + \dot r\hat\theta\dot\theta + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta + r\dot\theta(-\hat r)(\dot\theta)$

$= \ddot r\hat r + \dot r\dot\theta\hat\theta + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta - r\dot\theta^2\hat r$

$\vec a = (\ddot r - r\dot\theta^2)\hat r + (r\ddot\theta + 2\dot r\dot\theta)\hat\theta$

$\Rightarrow \vec a = (\ddot r - r\dot\theta^2)\hat r + (r\ddot\theta + 2\dot r\dot\theta)\hat\theta$

where $\ddot r - r\dot\theta^2$ is radial acceleration and $(r\ddot\theta + 2\dot r\dot\theta)$ is Angular Acceleration

29.01.06

The general form of the acceleration will be

$$\boxed{\vec{a} = a_r \hat{r} + a_\theta \hat{\theta}}$$

where $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

When $\dot{r} =$ Const Then $\ddot{r} = 0$ No radial Acceleration

$$(\text{As } \dot{\theta} = \tfrac{v}{r})$$

Then

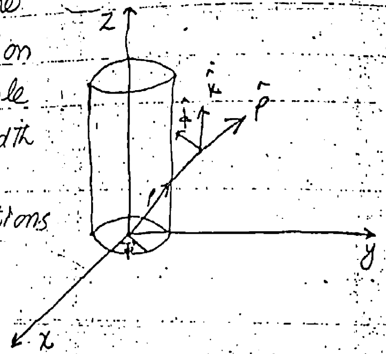
$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_r = (0) - r\dot{\theta}^2 = -r\left(\frac{v}{r}\right)^2 = -\frac{v^2}{r}$$

which is called Centripetal acceleration

* *

③ Spherical Cylendrical Coordinates :- (ρ, ϕ, z)



Consider a cylender in the plane, ϕ is the projection on the cylender making angle with x- axis, ρ is the width of the cylender.

The transformation equations are given as

$$x = \rho\cos\phi \ , \ y = \rho\sin\phi$$

$$z = z \ (\text{constant})$$

The inverse transformation as given below

$$\rho = \sqrt{x^2+y^2} \ , \ \phi = \tan^{-1} y/x \ , \ z = z$$

The Condition ρ varies from 0 to ∞

ϕ varies from 0 to 2π

z varies from $-z$ to z

$$\hat{\rho} = \hat{i}\cos\phi + \hat{j}\sin\phi$$

$$\hat{\phi} = -\hat{i}\sin\phi + \cos\phi\,\hat{j}$$

The Position vector $\vec{r}$ is given as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$$

$$= \rho(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + z\hat{k}$$

$$\vec{r} = \boxed{\rho\hat{\rho} + z\hat{k} \quad \text{— (A)}}$$

As $\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

29.01.06

The velocity will be given as

$$v = \frac{d}{dt}\vec{r} = \frac{d}{dt}(\rho\hat{\rho} + z\hat{k})$$

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \hat{k}\frac{dz}{dt} + z\frac{d\hat{k}}{dt}\xrightarrow{}0$$ since $\hat{k} =$ Const $\therefore \dot{\hat{k}} = 0$

$$= \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{d\phi}\frac{d\phi}{dt} + \dot{z}\hat{k}$$

$$\boxed{\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{k} \quad \text{— (B)}}$$

As $\dfrac{d\hat{\rho}}{d\phi} = \hat{\phi}$ and $\dfrac{d\phi}{dt} = \dot{\phi}$

The acceleration will be given as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{k})$$

$$= \ddot{\rho}\hat{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt} + \ddot{z}\hat{k} + \dot{z}\cancel{\frac{d\hat{k}}{dt}}^{0}$$

$$= \ddot{\rho}\hat{\rho} + \dot{\rho}\frac{d\hat{\rho}}{d\phi}\frac{d\phi}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{d\phi}\frac{d\phi}{dt} + \ddot{z}\hat{k}$$

$$= \ddot{\rho}\hat{\rho} + \dot{\rho}\hat{\phi}\dot{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}(-\hat{\rho})(\dot{\phi}) + \ddot{z}\hat{k}$$

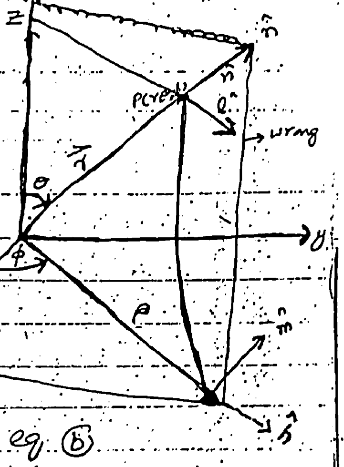
$$= \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} - \rho\hat{\rho}\dot{\phi}^2 + \ddot{z}\hat{k}$$

$$\boxed{\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{k} \quad \text{— (C)}}$$

This is the acceleration given above.

* *

④ Spherical polar Coordinates :- (r, θ, ϕ)



Let us consider a particle at $P(r,\theta,\phi)$ in the spherical polar coordinates, And $\hat{n}, \hat{e}, \hat{m}$ are the unit vectors in the increasing direction of r, θ and ϕ And $\vec{r}$ is the position vector then from figure

$$\hat{h} = \hat{i}\cos\phi + \hat{j}\sin\phi \quad \text{— ⓐ}$$

$$\hat{n} = \hat{k}\cos\theta + \hat{h}\sin\theta \quad \text{— ⓑ}$$

putting the value of $\hat{h}$ in eq ⓑ

$$\hat{n} = \hat{k}\cos\theta + (\hat{i}\cos\phi + \hat{j}\sin\phi)\sin\theta$$

$$\hat{n} = \hat{k}\cos\theta + \hat{i}\cos\phi\sin\theta + \hat{j}\sin\theta\sin\phi$$

29.01.06

$\hat{n} = \hat{k}\cos\theta + \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi$ —(c)

As $\hat{n}$ is $\perp$ to $\hat{l}$ Then we have

$\hat{l} = \hat{k}\cos(\theta+\pi/2)\, i\sin(\theta+\pi/2)\cos(\phi+\pi/2) + j\sin(\theta+\pi/2)\sin(\phi+\pi/2)$

$= -\hat{k}\sin\theta + i\cos\theta\cos\phi + j\cos\theta\sin\phi$

$= -\hat{k}\sin\theta + \cos\theta(\hat{i}\cos\phi + \hat{j}\sin\phi)$

$\hat{l} = -\hat{k}\sin\theta + \hat{h}\cos\theta$ —(d)

Similarly $\hat{m}$ is $\perp$ to $\hat{h}$

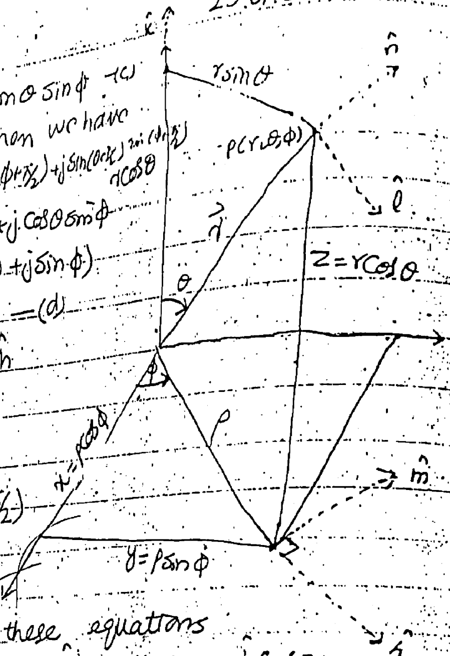
we have

$\hat{h} = \hat{i}\cos\phi + \hat{j}\sin\phi$

$\hat{m} = i\cos(\phi+\pi/2) + j(\sin(\phi+\pi/2))$

$= -\sin\phi\,\hat{i} + \hat{j}\cos\phi$

$\hat{m} = -\hat{i}\sin\phi + \hat{j}\cos\phi$ — ⓔ



Hence we have all these equations

$\hat{h} = \hat{i}\cos\phi + \hat{j}\sin\phi$, $\quad \hat{l} = -\hat{k}\sin\theta + \hat{h}\cos\theta$

$\hat{n} = \hat{k}\cos\theta + \hat{h}\sin\theta$, $\quad \hat{m} = -\hat{i}\sin\phi + \hat{j}\cos\phi$

Taking the derivatives of above eq w.r.t θ & ϕ we have

$$\frac{d\hat{h}}{d\phi} = \hat{m}, \quad \frac{d\hat{m}}{d\phi} = -\hat{h}, \quad \frac{d\hat{h}}{d\theta} = 0 = \frac{d\hat{m}}{d\theta}$$

$$\frac{d\hat{l}}{d\theta} = -\hat{n}, \quad \frac{d\hat{n}}{d\theta} = \hat{l}$$

$$\frac{d\hat{n}}{d\phi} = \hat{m}\sin\theta, \quad \frac{d\hat{l}}{d\phi} = \hat{m}\cos\theta$$

From The fig $\quad \vec{\rho} = x\hat{i} + y\hat{j}$

$\Rightarrow \quad \rho = \sqrt{x^2+y^2}$

$\Rightarrow \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

In spherical polar coordinates we have

$x = r\cos\phi\sin\theta$, $\quad y = r\sin\theta\sin\phi$, $\quad Z = r\cos\theta$

$x^2+y^2+z^2 = (r\cos\phi\sin\theta\,\hat{i})^2 + (r\sin\theta\sin\phi\,\hat{j})^2 + (r\cos\theta\,\hat{k})^2$

$x^2+y^2+z^2 = r^2\cos^2\phi\sin^2\theta + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$

$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta$

$x^2+y^2+z^2 = r^2(\sin^2\theta + \cos^2\theta)$

$x^2+y^2+z^2 = r^2$

$\Rightarrow \quad r = \sqrt{x^2+y^2+z^2}$

29.01.06

From The Figure. $\tan\theta = \rho/z = \sqrt{x^2+y^2}/z \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$\tan\phi = y/x \quad$ or $\quad \phi = \tan^{-1}(y/x)$

or $\phi = \cos^{-1}\frac{x}{\rho}$, $\quad \phi = \sin^{-1}(y/\rho)$

Now we going to find The position vector in term of $\hat{l}$, $\hat{m}$ and $\hat{n}$ we have

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

$= r\sin\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + r\cos\theta\,\hat{k}$

$= r\sin\theta(\hat{h}) + r\cos\theta\,\hat{k}$

$= r(\hat{h}\sin\theta + \cos\theta\,\hat{k})$

$\vec{r} = r\,\hat{n}(\theta,\phi)$ — Ⓘ

The equation (I) gives The position vector in The spherical coordinate. The velocity will be given as

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}r\hat{n}$$

$$= \dot{r}\hat{n} + r\frac{d\hat{n}}{dt} \quad \text{because } \hat{n} = \hat{n}(\theta,\phi)$$

$$= \dot{r}\hat{n} + r\left[\frac{d\hat{n}}{d\phi}\cdot\frac{d\phi}{dt} + \frac{d\hat{n}}{d\theta}\cdot\frac{d\theta}{dt}\right]$$

$$= \dot{r}\hat{n} + r\left[\dot{\phi}\,\hat{m}\sin\theta + \dot{\theta}\,\hat{l}\right]$$

$$\boxed{V = \dot{r}\hat{n} + r\dot{\theta}\hat{l} + r\dot{\phi}\,\hat{m}\sin\theta} \text{ — II}$$

The acceleration will be given as

$$\vec{a} = \frac{d}{dt}\vec{V} = \ddot{r}\hat{n} + \dot{r}\frac{d\hat{n}}{dt} + \dot{r}\dot{\theta}\hat{l} + r\ddot{\theta}\hat{l} + r\dot{\theta}\frac{d\hat{l}}{dt} + \dot{r}\dot{\phi}\hat{m}\sin\theta + r\ddot{\phi}\hat{m}\sin\theta + r\dot{\phi}\sin\theta\frac{d\hat{m}}{dt} + r\dot{\phi}\hat{m}\frac{d}{dt}\sin\theta$$

we putt The values in eq above as such

$$\frac{d\hat{n}}{dt} = \frac{d\hat{n}}{d\theta}\cdot\frac{d\theta}{dt} + \frac{d\hat{n}}{d\phi}\cdot\frac{d\phi}{dt} = \dot{\theta}\hat{l} + \dot{\phi}\sin\theta\,\hat{m} \text{ —(a)}$$

$$\frac{d\hat{l}}{dt} = \frac{d\hat{l}}{d\theta}\cdot\frac{d\theta}{dt} + \frac{d\hat{l}}{d\phi}\cdot\frac{d\phi}{dt} = -\dot{\theta}\hat{l} + \dot{\phi}\cos\theta\,\hat{m} \text{ —(b)}$$

$$\frac{d\hat{m}}{dt} = \frac{d\hat{m}}{d\phi}\cdot\frac{d\phi}{dt} = -\dot{\phi}\,\hat{n}\sin\theta - \cos\phi\,\dot{\phi}\,\hat{l} \text{ —(c)}$$

we get

$\vec{a} = \ddot{r}\hat{n} + \dot{r}(\dot{\theta}\hat{l} + \dot{\phi}\sin\theta\hat{m}) + \dot{r}\dot{\theta}\hat{l} + r\ddot{\theta}\hat{l} + r\dot{\theta}(-\dot{\theta}\hat{n} + \dot{\phi}\cos\theta\hat{m})$

$\vec{a} = \ddot{r}\hat{n} + \dot{r}(\dot{\theta}\hat{l} + \dot{\phi}\sin\theta\hat{m}) + \dot{r}\dot{\theta}\hat{l} + \dot{r}\dot{\phi}\sin\theta\hat{m} + r\ddot{\phi}\sin\theta\hat{m} + r\dot{\phi}\dot{\theta}\cos\theta\hat{m} - r\dot{\phi}^2\sin\theta\cos\theta\hat{l}$
$+ \dot{r}\dot{\phi}\sin\theta\hat{m} + r\dot{\phi}\sin\theta\hat{m} + r\dot{\phi}\sin\theta\hat{m}$

$\Rightarrow \vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{n} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{l}$
$+ (r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta)\hat{m}$

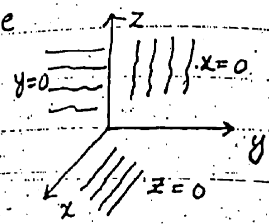
$\Rightarrow \boxed{\vec{a} = a_r\hat{n} + a_\theta\hat{l} + a_\phi\hat{m}}$

This is the required acceleration

* *

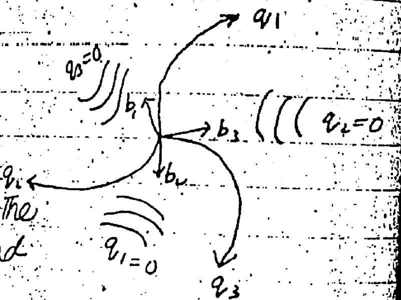
Curvilinear (Generalized) Coordinate System :-

A suitable set of surfaces i.e $x=0, y=0, z=0$ can be used as a reference axis and intersection of these surfaces as a reference axis.



Let us denote the curved coordinates by q_1, q_2 and q_3 respectively.

It should be noted that q_1 axis is an intersection of the two surfaces q_2 = constant and q_3 = constant



Cartesian Coordinates are related to it by

$$\left.\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned}\right\} — ①$$

The inverse Transformations are

$$\left.\begin{aligned} q_1 &= q_1(x, y, z) \\ q_2 &= q_2(x, y, z) \\ q_3 &= q_3(x, y, z) \end{aligned}\right\} — ②$$

The base vectors b_1, b_2 and b_3 are the vectors drawn in the direction of change of position vector $\vec{r}$ produced by an infinitismal change in the coordinates q_1, q_2 and q_3 respectively when the other coordinates are fixed

then $\vec{b_i} = \frac{\partial r_i}{\partial q_i}$ — ③ where $i = 1, 2, 3 \ldots$

In general base vector need not to be a unit vector It can be related to the unit vector(sys) $\hat{e}_i$ by where $\hat{e}_i$ is the unit vector

$$\vec{b_i} = h_i\hat{e}_i \quad —①$$

Where h_i is a scalar factor whose value $h_i = \left|\frac{\partial r_i}{\partial q_i}\right|$

Then eq ④ can be written as

$$\hat{e}_i = \frac{1}{h_i} b_i = \frac{\partial r_i/\partial q_i}{|\partial r_i/\partial q_i|} \quad —⑤$$

In term of Cartesian Coordinates, the unit vectors are $\hat{i}, \hat{j}$ and $\hat{k}$ Then

$$\vec{b_i} = h_i\hat{e}_i$$

$$\Rightarrow \vec{b_i} = \frac{\partial x}{\partial q_i}\hat{i} + \frac{\partial y}{\partial q_i}\hat{j} + \frac{\partial z}{\partial q_i}\hat{k}$$

This means that $h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}$ —⑥

For Cartesian Coordinates $q_1 = x,\ q_2 = y$ & $q_3 = z$

When $i = 1$ Then

$$b_i = \frac{\partial x}{\partial q_1}\hat{i} + \frac{\partial y}{\partial q_1}\hat{j} + \frac{\partial z}{\partial q_1}\hat{k} = \frac{\partial x}{\partial x}\hat{i} + \overset{0}{\cancel{\frac{\partial y}{\partial x}}}\hat{j} + \overset{0}{\cancel{\frac{\partial z}{\partial x}}}\hat{k} = \hat{i}$$

When $i = 2$ Then

$$b_i = \frac{\partial x}{\partial q_2}\hat{i} + \frac{\partial y}{\partial q_2}\hat{j} + \frac{\partial z}{\partial q_3}\hat{k} = \overset{0}{\cancel{\frac{\partial x}{\partial y}}}\hat{i} + \frac{\partial y}{\partial y}\hat{j} + \overset{0}{\cancel{\frac{\partial z}{\partial y}}}\hat{k} = \hat{j}$$

When $i = 3$ Then

$$b_i = \frac{\partial x}{\partial q_3}\hat{i} + \frac{\partial y}{\partial q_3}\hat{j} + \frac{\partial z}{\partial q_3}\hat{k} = \overset{0}{\cancel{\frac{\partial x}{\partial z}}}\hat{i} + \overset{0}{\cancel{\frac{\partial y}{\partial z}}}\hat{j} + \frac{\partial z}{\partial z}\hat{k} = \hat{k}$$

And

$$b_i = \frac{\partial \bar{r}}{\partial q_i} = \frac{\partial}{\partial q_i}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$b_i = \frac{\partial x}{}\hat{i} + \frac{\partial y}{}\hat{j} + \frac{\partial z}{}\hat{k}$$

In order to define a coordinate system the three surfaces $q_1 =$ Constt, $q_2 =$ Constt and $q_3 =$ Constt must obviously be non coplanner.

∴ (should not be having the same plane)

This means that the base vector must be non coplanner, the condition for this is for $i = 1,2,3$

Then

$$(b_1, b_2, b_3) = \begin{vmatrix} \frac{\partial x}{\partial q_1}, & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \partial y/\partial q_2 & \frac{\partial z}{\partial q_2} \\ \frac{\partial x}{\partial q_3} & \partial y/\partial q_3 & \partial z/\partial q_3 \end{vmatrix} \neq 0 \quad ⑦$$

Non Coplaner

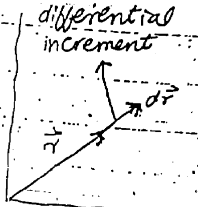
This determinent is known as Jacobian and the condition is putt into a compact form or compact notation.

$$\frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} \neq 0 \quad — \quad \text{(7-A)}$$

The distance between two neighbouring points in Cartesian Coordinate System is

$|d\vec{r}|^2 = dr^2 = dx^2 + dy^2 + dz^2$

(As: $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$)



The differential increments dx, dy and dz in x, y and z are related to the increments q_1, q_2 and q_3 through usual mathamatical rules

$x = x(q_1, q_2, q_3)$

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 \quad ⑧$$

→ Partial diff rules

Similarly $y = y(q_1, q_2, q_3)$

$$dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3 \quad ⑧$$

$z = z(q_1, q_2, q_3)$

$$dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3$$

For only x-component we have

$$dr^2\Big|_x = \left(\frac{\partial x}{\partial q_1}\right)^2 dq_1^2 + \left(\frac{\partial x}{\partial q_2}\right)^2 dq_2^2 + \left(\frac{\partial x}{\partial q_3}\right)^2 dq_3^2$$

In general we have for all components

$$dr^2 = \left(\frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} dq_i dq_j + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} dq_i dq_j + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} dq_i dq_j\right)$$

Where $i = j = 1, 2, 3 \ldots$

For $i = 1$ and $j = 1$ putt in above we have

$$dr^2 = \left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_1} dq_1 dq_1 + \frac{\partial y}{\partial q_1}\frac{dy}{\partial q_1} dq_1 dq_1 + \frac{\partial z}{\partial q_3}\frac{\partial z}{\partial q_1} dq_3 dq_2\right)$$

For $j=2$ $$+\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} dq_1 dq_2 + \frac{\partial x}{\partial q_2}\frac{\partial x}{\partial q_2} dq_2 dq_2 + \frac{\partial x}{\partial q_3}\frac{\partial x}{\partial q_2} dq_3 dq_2$$

For $j=3$ $$+\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_3} dq_1 dq_3 + \frac{\partial x}{\partial q_2}\frac{\partial x}{\partial q_3} dq_2 dq_3 + \frac{\partial x}{\partial q_3}\frac{\partial x}{\partial q_3} dq_3 dq_1$$

Where $$h_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$$

$$\Rightarrow dr^2 = \sum_{i,j} h_{ij}\, dq_i dq_j \quad ⑨$$

Where

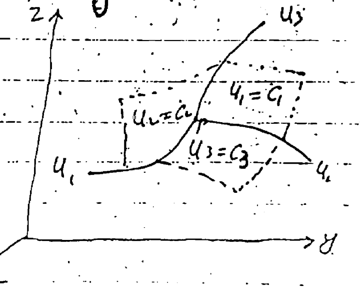
$$h_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$$

The h_{ij} is called matric and specify the nature of the coordinates, when one cannot express the function / line / number in terms of the other.

* *

Orthogonal Curvilinear Systems:-

The surfaces defined by $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ where, c_1, c_2 and c_3 are constants, are called coordinate surfaces and each pair of these surfaces intersect in curves



29.01.06

called coordinate curves or lines. If the coordinate surfaces intersect at rigth angle to the curvilinear coordinates system is called orthognal the u_1, u_2 and u_3 coordinates curves of a curvilinear system are analogous to the x, y and z coordinate axes of a rectangular system Hence forth we shall always refer to orthognal curvilinear coordinates unless otherwise stated

For coordinate system to be orthognal then the unit vectors must specify the condition

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{bmatrix} 0 & i \neq j \\ 1 & i = j \end{bmatrix} \rightarrow \text{Kronecter's Delta}$$

This is the condition of orthognality, then the distance between the two neighbouring points

$$dr^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

where $\sqrt{h_{11}} = h_1$, $\sqrt{h_{22}} = h_2$ & $\sqrt{h_{33}} = h_3$

The cross terms are zero or missing.

$$d\vec{r} = \vec{b}_1 dq_1 + \vec{b}_2 dq_2 + \vec{b}_3 dq_3$$

putting the values of b_1, b_2 and b_3

$$d\vec{r} = \hat{e}_1 h_1 dq_1 + \hat{e}_2 h_2 dq_2 + \hat{e}_3 h_3 dq_3$$

$$\Rightarrow d\vec{r} = \sum \hat{e}_i h_i dq_i \qquad (\text{As } b_1 = \hat{e}_1 h_1)$$

This is the general form.

the $$d\vec{r}^2 = h_1^2 (dq_1)^2 + h_2^2 (dq_2)^2 + h_3^2 (dq_3)^2$$

$$d\vec{r}^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

The three curvilinear coordinates q_1, q_2 and q_3 need not be length. The scale factors h_i (is a function) may depends on q_3. The product $h_i dq_i$ must have the dimension of length.

* *

29.01.06

In cartesian coordinates :-

$h_1 = 1$, $h_2 = 1$, $h_3 = 1$ $(q_1 = x, q_2 = y, q_3 = z)$

For plane Polar coordinates :-

$q_1 = r$, $q_2 = \theta$ $(\hat{e}_r, \hat{e}_\theta)$

$h_1 = 1$, $h_2 = r$

For cylenderical coordinates :-

$q_1 = \rho$, $q_2 = \phi$, $q_3 = z$

$h_1 = 1$, $h_2 = \rho$, $h_3 = 1$

For spherical polar coordinates :-

$q_1 = r$, $q_2 = \theta$, $q_3 = \phi$

$h_1 = 1$, $h_2 = r$, $h_3 = h\phi = r\sin\theta$

* *

CHAP NO. 1

Gold stein

Survey of the Elementary Principles

CHAPTER NO. ③ RG TAKWALE # 66

MECHANICS OF A PARTICLE :-

In mechanics the problem is solved and developed with the help of Newton 2nd law. The 2nd order differential equation is

$$F = m \frac{d^2r}{dt^2} \quad —①$$

This is solved with the help of initial value problem / Boundry value problem. The Force is

$$\bar{F} = \bar{F}(\vec{r}, v, t)$$
$$= \bar{F}(\vec{r}, \dot{\vec{r}}, t)$$

Force is applied and depends on different variables. The solution of eq ① is

$$\vec{r} = \vec{r}(t)$$

$$F = m\, dv/dt$$

If a is uniform then

$$F = ma = m\frac{dv}{dt} \Rightarrow a = dv/dt$$

30.01.06

$$a\,dt = dV$$

$$\Rightarrow a\int dt = \int dV$$

Initial Condition: when $t=0$ Then $V=V_0$
when $t=t$ Then $V=V$

$$a\int_0^t dt = \int_V dV$$

$$V = at + V_0 \quad \text{—(A)}$$

When $t=0$ Then $V=V_0$

$$V_0 = C$$

Then $\boxed{V = at + V_0}$

Similarly $V = \frac{dx}{dt} = at + V_0$

$$\Rightarrow dx = (at + V_0)dt$$

$$\int_{x_0}^{x} dx = a\int_0^t t\,dt + V_0\int_0^t dt$$

When $t=0$ Then $x = x_0$ So The Condition

$$x = \frac{at^2}{2} + V_0 t + D$$

Apply Condition $x_0 = D$ Then

$$x = \frac{at^2}{2} + V_0 t + x_0$$

or $x - x_0 = \frac{1}{2}at^2 + V_0 t$

* * *

(I) Conservation For Linear momentum:-

If no external force is acting on a Particle Then The $F=0$ Then $F = \frac{dp}{dt} = 0$

$$\Rightarrow dp = 0 \times dt$$

$$\int dp = \int 0$$

$$\Rightarrow P = \text{Constt}$$

which means that when no external force is acting on a Particle Then its linear momentum is Conserved

* * *

MATI ULLAH
M.Sc (Physics)
0345-9798233
BANNU

30.01.06

(II) Conservation of The angular momentums:-

The angular momentum of a Particle at a Point 'o' is given as $\bar{L} = \bar{r} \times \bar{P}$
where $\bar{r}$ is The Position vector and $\bar{P}$ is The linear momentum for The Particle
Then The Torque (Turning effect) of the body is defined as

$$\vec{N} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{P})$$

$$= \dot{\vec{r}} \times \vec{P} + \vec{r} \times \dot{\vec{P}}$$

$$= (\overset{\to 0}{\dot{\vec{r}} \times m\vec{v}}) + \vec{r} \times \frac{d\vec{P}}{dt} \quad (\text{As } \dot{\vec{P}} = \frac{dP}{dt} = F)$$

$$\vec{N} = \vec{r} \times \vec{F}$$

If no Torque is acting means Torque is zero then

$$0 = \vec{N} = \frac{dL}{dt} = \bar{r} \times \vec{F} = 0$$

$$\Rightarrow \frac{dL}{dt} = 0 \Rightarrow dL = 0$$

$$\int dL = \int 0$$

$$\Rightarrow L = \text{Constt}$$

It means That if The No Torque is acting on a body Then The angular momentum will be Conserved.
For exemple rotation of Planets about The Sun is The most good Exemple.

WORK DONE

(III) The work done by The total external force in moving a particle from Position 1 to position 2 is given by

$$W_{1\to 2} = \int_1^2 \vec{F}_{ext} \cdot d\vec{r} = \int_1^2 \frac{dp}{dt} \cdot d\vec{r}$$

$$= \int_1^2 \frac{d}{dt} mv \cdot d\vec{r} = m\int_1^2 \frac{dv}{dt} d\vec{r}$$

$$= m\int_1^2 \left(\frac{dv}{dt} \cdot \frac{d\vec{r}}{dt}\right)dt = m\int_1^2 \left(\frac{dv}{dt} \cdot v\right)dt$$

01.02.06

$\vec{P}_i = F_i^{(e)} + \sum_j' F_{ji}$ As $\vec{P} = ma$

So putting value of $\vec{P}_i$ we get $\vec{P} = m_i \frac{d^2\vec{r}_i}{dt^2}$

$m_i \frac{d^2\vec{r}_i}{dt^2} = F_i^{(e)} + \sum_j' F_{ji}$ — ① As $F_{ij} = -F_{ji}$ Then $F_{ij} + F_j = 0$

The second term in eq ① vanishes as So 2nd Term on eq L.H.S vani

$F_{ji} + F_{ij} = 0$, $\sum F_{ji} = 0$.

$m_i \frac{d^2\vec{r}_i}{dt^2} = F_i^{(e)} + 0$ — ②

To reduce the L.H.S of the above equation we define a vector $\vec{R}$, as the average of the radii vectors of the particles weighted in proportion to their masses

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum m_i}$$

where $\vec{R}$ is the center of mass

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum m_i} = \frac{\sum_i m_i \vec{r}_i}{M} \quad (\text{As } \sum m_i = M)$$

The vector $\vec{R}$ defines a point known as the Center of mass. Then eq ② ⇒

$$\sum_i m_i \frac{d^2 r_i}{dt^2} = \sum_i F_i^{(e)} + \cancel{\sum_{ij}' F_{ji}}^{\,0}$$

$$\sum_i m_i \frac{d^2 r_i}{dt^2} = \sum_i F_i^{(e)} \qquad R = \frac{\sum m_i r_i}{\sum m_i}$$

So putting value of R.H.S $R = \frac{\sum m_i r_i}{M}$

$$M \frac{d^2 R}{dt^2} = \sum_i F_i^{(e)}$$

$$MR = \sum_i m_i r_i$$

$$M \frac{d^2 R}{dt^2} = \sum_i m_i \frac{d^2 r_i}{dt^2}$$

$$\Rightarrow \boxed{\sum_i F_i^{(e)} = M \frac{d^2 R}{dt^2}}$$

$$\Rightarrow \boxed{\vec{F}^{(e)} = M\ddot{\vec{R}}}$$

01.02.06

which states that center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass, where

M is the total mass of the system

$\vec{R}$ position vector of the center of mass of system

* * *

Conservation Theorem for the Linear Momentum:-

The Conservation theorem for the linear momentum of a system of particle say that if the total external force is zero, then the linear momentum is conserved.

The total linear momentum of the system is

$$\vec{P} = \sum_i \vec{P}_i = \sum_i m_i \frac{dr_i}{dt} = M \frac{dR}{dt}$$

$$F = \frac{dp}{dt}$$

This is the total linear momentum of the system which is the total mass of the system times the velocity of the center of mass.

If no external force is acting then

$$\dot{\vec{P}} = 0$$

$$\int \dot{\vec{P}} = \int 0 \Rightarrow \vec{P} = \text{Const}$$

which give the result that when total external force is zero then the linear momentum is conserved.

* *

~~of~~ Angular Momentum of The System of Particles :-

OR

Conservation Theorem For Total Angular Momentum :-

The total angular momentum will be vector sum of the individual angular momentum of individual particles about any point.

01.02.06

$\vec{P}_i = F_i^{(e)} + \sum_j' F_{ji}$ As $\vec{P} = ma$

so putting value of $\vec{P}_i$ we get $\vec{P} = m_i \frac{d^2\vec{r}_i}{dt^2}$

$$m_i \frac{d^2\vec{r}_i}{dt^2} = F_i^{(e)} + \sum_j' F_{ji} \quad \text{—①}$$

As $F_{ij} = -F_{ji}$ Then $F_{ij} + F_{ji} = 0$

The second term in eq① vanishes as $F_{ji} + F_{ij} = 0$

So 2nd Term on eq① L.H.S vanish

$$m_i \frac{d^2\vec{r}_i}{dt^2} = F_i^{(e)} + 0 \quad \text{—②} \qquad \sum_j F_{ji} = 0$$

To reduce the L.H.S of the above equation we define a vector $\vec{R}$, as the average of the radii vectors of the particles weighted in proportion to their masses

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum m_i}$$

Where $\vec{R}$ is the center of mass

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M} \quad (\text{As } \sum m_i = M)$$

The vector $\vec{R}$ defines a point known as the Center of mass then eq②⇒

$$\sum_i m_i \frac{d^2 r_i}{dt^2} = \sum_i F_i^{(e)} + \cancelto{0}{\sum_{ij}' F_{ji}}$$

$$\sum_i m_i \frac{d^2 r_i}{dt^2} = \sum_i F_i^{(e)} \qquad R = \frac{\sum m_i r_i}{\sum m_i}$$

So putting value of R.H.S $\quad R = \frac{\sum m_i r_i}{M}$

$$M \frac{d^2R}{dt^2} = \sum_i F_i^{(e)} \qquad MR = \sum_i m_i r_i$$

$$M \frac{d^2R}{dt^2} = \sum_i m_i \frac{d^2 r_i}{dt^2}$$

$$\Rightarrow \boxed{\sum_i F_i^{(e)} = M \frac{d^2R}{dt^2}} \Rightarrow \boxed{\vec{F}^{(e)} = M\ddot{\vec{R}}}$$

01.02.06

which states that center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass where

M is the total mass of the system

$\vec{R}$ position vector of the center of mass of system

* * *

Conservation Theorem for the Linear Momentum:-

The conservation theorem for the linear momentum of a system of particle say that if the total external force is zero, then the linear momentum is conserved.

The total linear momentum of the system is

$$\vec{P} = \sum_i \vec{P}_i = \sum_i m_i \frac{dr_i}{dt} = M \frac{dR}{dt}$$

$F = \frac{dp}{dt}$

This is the total linear momentum of the system which is the total mass of the system times the velocity of the center of mass.

If no external force is acting then

$$\dot{\vec{P}} = 0$$

$$\int \dot{\vec{P}} = \int 0 \Rightarrow \vec{P} = \text{Const}$$

which give the result that when total external force is zero then the linear momentum is conserved.

* *

Angular Momentum of the System of Particles:-

OR

Conservation Theorem For Total Angular Momentum:-

The total angular momentum will be vector sum of the individual angular momentum of individual particles about any point.

If $\vec{l_i}$ represent angular momentum of ith particles, Then $l_i = r_i \times p_i$ where r_i is The position vector of ith particle from a given Point the Total angular momentum will be

$$L = \sum_i l_i = \sum_i (r_i \times p_i)$$

Let N be total Torque acting on system

$$N = \frac{\partial L}{\partial t} = \sum_i \frac{\partial}{\partial t}(r_i \times p_i)$$

$$= \sum_i (\dot{r}_i \times p_i) + \sum_i (\dot{\vec{P}}_i \times \vec{r})$$

Here $\sum_i (\dot{r}_i \times p_i) = \sum_i (\dot{r}_i \times m_i \dot{\vec{r}}_i) = 0$

Then $N = \sum_i (\dot{\vec{P}}_i \times \vec{r}_i)$

$$N = \sum_i (\dot{\vec{P}}_i \times \vec{r}_i)$$

As $\dot{\vec{P}}_i = \vec{F}^{(e)} + \sum_{i,j} F_{ij}$

$$N = \sum_i \left[\vec{r}_i \times \left(\vec{F}^{(e)} + \sum_{i,j} F_{ij}\right)\right]$$

For entire system $\sum_{i,j} F_{ij} = 0$ Then

$$N = \sum_i (\vec{r}_i \times \vec{F}^{(e)})$$

$$\Rightarrow \vec{N} = \sum_i (\vec{r}_i \times \vec{F}^{(e)})$$

✓ If Total Torque N = 0 Then

$$N = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow dL = 0$$

$$\int dL = \int 0 \Rightarrow L = \text{Constt}$$

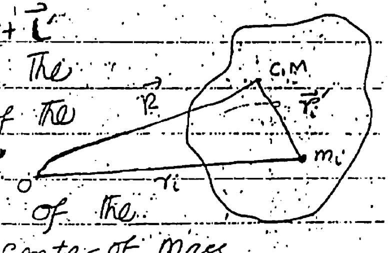
which means That when The Total

Torque is zero Then The angular momentum for The System of particles will be conserved.

◆

Prove That The angular momentum of The System about The fixed point is equal to The sum of The angular momentum of The total mass Concentrated at The center of mass about That Point and The angular momentum of The system about it's Center of Mass Mathematically: $\vec{L} = (R \times \vec{P})_{c.m} + \vec{L}'$

Proof:- $\vec{L} = \vec{L}_{c.m} + \vec{L}'$



Let $\vec{m}_i$ and $\vec{r}_i$ represent The mass and Position vector of The ith Particle with respect to 'O'. $\vec{r}_i'$ is The Position vector of The ith particle w.r.t to center of mass. $\vec{R}$ is The Position vector of The center of mass about "O" Then The Transformation Eq will be

$$r_i = \vec{R} + \vec{r}_i'$$

The corresponding velocities will be

$$\dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{r}}_i'$$

or $\vec{v} = \vec{V} + \vec{v}_i'$

$\vec{V}$ = velocity of The center of mass

$\vec{v}$ = velocity of ith Particle w.r.t to c.m

$\vec{v}'$ = velocity of ith Particle w.r.t to 'O'

The total angular momentum will be

$$L = \sum_i (\vec{r}_i \times \vec{P}_i)$$

$$= \sum_i (\vec{R} + \vec{r}_i') \times \sum_i \vec{P}_i$$

$$= \sum_i (\vec{R} + \vec{r}_i') \times \sum m_i \dot{\vec{r}}_i$$

$$= \sum_i (\vec{R} + \vec{r}_i') \times \sum m_i (\dot{\vec{R}} + \dot{\vec{r}}_i')$$

$$L = \sum_i (\vec{R} * \vec{r}_i') \times \sum_i (\vec{R} + \vec{r}')\, m_i$$

$$= \sum_i (\vec{R} + \vec{r}_i') * \sum_i (m_i \dot{R} + m_i \dot{\vec{r}}')$$

$$L = \sum_i (\vec{R} \times m_i \dot{\vec{R}}) + \sum_i (\vec{R} \times m_i \dot{\vec{r}}') + \sum_i (\vec{r}_i' \times m_i \dot{\vec{R}}) + \sum_i (\vec{r}_i' \times m_i \dot{\vec{r}}')$$

The second and 3rd Term vanishes

$$\because \sum_i (\vec{R} \times m_i \dot{\vec{r}}') = R \times \sum_i m_i \dot{\vec{r}}_i' = R \times \frac{d}{dt} \sum_i m_i r_i' = 0$$

The $\vec{P}_i'$ is measured w.r. to same frame of reference $\therefore$ coplanner vectors have no result.

$$\vec{L} = \sum_i (\vec{R} \times m_i \dot{\vec{R}}) + \sum_i (\vec{r}_i' \times m_i \dot{\vec{r}}')$$

$$= \vec{R} \times \dot{\vec{R}} \sum_i m_i + \sum_i (\vec{r}_i' \times \vec{P}_i')$$

$$= \vec{R} \times M\dot{\vec{R}} + \sum_i (\vec{r}_i' \times \vec{P}_i')$$

$$\therefore \boxed{\vec{L} = \vec{L}_{cm} + \vec{L}'}$$

Angular momentum about The Center of Mass

$$\boxed{L = (R \times P_{cm}) + \vec{L}'}$$

ENERGY OF THE SYSTEM :-

The energy of The system is determined by the work done by all the forces (external force as well as internal forces) in moving The system from some initial Configuration to some final Configuration thus The work done will be

02.02.06

$$W_{12} = \sum_i \left[\int_1^2 F_i \cdot d\bar{r}_i\right] = \sum_i \left[\int_1^2 \bar{F}_i^{(e)} \cdot dr_i + \sum_{i,j}' \int F_{ji}^{(int)} \cdot dr_i\right]$$

$$W_{1\to 2} = \sum \left[\int_1^2 F_i^{(e)} \cdot dr_i + \sum_{i,j}' \int_1^2 \vec{F}_{ji}^{(int)} \cdot d\bar{r}_i\right] — ①$$

If all the forces are conservative (internal as well as external), Then

$$F_i = F_i^{(e)} + \sum_j \vec{F}_{ji}^{(int)} = -\bar{\nabla}_i V_i — ②$$

Where The Potential Energy has again Two Parts

$$V_i = V_i^{(e)} + V_i^{(int)}$$

and

$$\bar{\nabla}_i = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} + \dots \hat{e}_n \frac{\partial}{\partial x_n}$$

↳ This represent The gradient operator performing differential with respect to x_i Component of Position vector $\bar{r}_i$ of ith Particle.

In eq ② we can represent seperately as,

$$\bar{F}_i^{(e)} = -\nabla_i V_i \; , \quad \bar{F}_{ji}^{(int)} = -\bar{\nabla}_{ji} V_{ji}^{(int)} — ②a$$

Where $V_{ji}^{(int)}$ is The Potential Energy arising due to internal forces $\vec{F}_{ji}^{(int)}$ and

$$\bar{\nabla}_{ji} = \hat{e}_i \frac{\partial}{\partial (x_j - x_i)} \qquad \vec{\nabla}_{12} = \hat{e}_1 \frac{\partial}{\partial (x_1 - x_2)}$$

and

$$\bar{\nabla}_{ji} = -\bar{\nabla}_{ij} \qquad \vec{\nabla}_{21} = -\hat{e}_1 \frac{\partial}{\partial (x_2 - x_1)}$$

Hence The Total P.E due to internal forces is

$$V^{(int)} = \sum_i V_i^{(int)} = \sum_{i,j}' V_{ji}^{int} — ③ \quad (i<j)$$

$$(V_{ii} = V_{jj} = 0)$$

Self interaction is not Considered and $i<j$ to avoide The double Terms.

02.02.06

By Newton 3rd law we have eq (2a) ⇒

$\bar{F}_{ji} = -\bar{\nabla}_{ji} V_{ji} = -\bar{F}_{ij} = \nabla_{ij} V_{ij}$ — (4)

The terms

$-\bar{\nabla}_{ji} V_{ji} = \bar{\nabla}_{ij} V_{ij}$

$~~-\bar{\nabla}_{ji} V_{ji} = -\bar{\nabla}_{ji} V_{ij}~~$

$\Rightarrow V_{ji} = V_{ij}$

Thus the Equation (3) can be written as

$$V^{(int)} = \sum_i V_i^{(int)} = \frac{1}{2} \sum_{ij}{}' V_{ij}^{int} \quad — (5)$$

The work done by the External force is

$\left.\begin{array}{l}\frac{1}{2}(V_{11}+V_{12}+V_{21}+V_{22} \\ = \frac{1}{2}(2V_{12}) = V_{12}\end{array}\right\}$

$$W^{(ext)} = \sum_i \int_1^2 \bar{F}_i^{(e)} \cdot d\bar{r}_i = -\sum \int_1^2 \bar{\nabla}_i V_i^{ext} dr_i$$

$\left.\right\}$ As $\nabla_i V_i dr_i = dV_i$

$\bar{\nabla}_i$ = when external force is derivable from the scalar function V

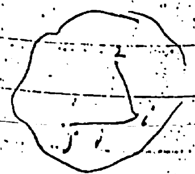
$$\sum_i \int_1^2 F_i^{(e)} \cdot dr_i = -\sum_i \int_1^2 dV_i^{ext} = -\sum_i V_i^{(ext)}\Big|_1^2$$

$$= -(V_2^{(ext)} - V_1^{(ext)}) = V_1^{(ext)} - V_2^{(ext)} \quad — (6)$$

The work done by the internal force is

$$\sum_{i,j} \int_1^2 F_{ji}^{(int)} \cdot d\vec{r}_j = \sum_{i,j}{}' \int_1^2 F_{ij} \cdot d\bar{r}_i$$

As work is a scalar quantity. Therefore both are equal not depends on i, j



$$= \sum_{i,j}{}' F_{ij} \cdot d\bar{r}_i = -\sum_{i,j}{}' F_{ji} \cdot d\bar{r}_i$$

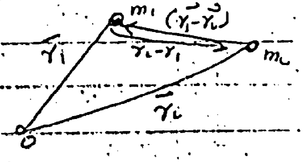
∴ Then

$$\sum_{i,j}{}' \int_1^2 \vec{F}_{ji} \cdot d\bar{r}_j = \frac{1}{2} \int_1^2 \sum_{i,j} F_{ij} (dr_j - d\bar{r}_i)$$

$$= \frac{1}{2} \int_1^2 \sum_{i,j}{}' \vec{F}_{ij} (d\bar{r}_i - d\bar{r}_j)$$

02.02.06

$$= \frac{1}{2} \sum_{(i,j)}{}' \int_1^2 \bar{F}_{ji} \cdot d\bar{r}_{ji}$$



Thus

$$\sum_{(i,j)}{}' \int_1^2 F_{ji} \cdot dr_j = \frac{1}{2} \sum_{(i,j)}{}' \int_1^2 \bar{F}_{ji} \cdot dr_{ji}$$

$$= \frac{1}{2} \sum_{(i,j)}{}' \int_1^2 (-\nabla_{ji} V_{ji}) \cdot d\bar{r}_{ji} \quad (\text{As } \nabla V \cdot dr = dV)$$

$$= -\frac{1}{2} \sum_{(i,j)}{}' \int_1^2 dV_{ji}^{(int)} = -\frac{1}{2} \sum_{(i,j)}{}' V_{ji}^{int}\Big|_1^2$$

From eq (5) we have

$$V^{int} = -[V^{int}]_1^2 \Rightarrow V_1^{int} - V_2^{int} \quad — (7)$$

The total P.E of the system is

$$V = V^{ext} + V^{int}$$

$$= (V_1^{ext} - V_2^{ext}) + (V_1^{int} - V_2^{int})$$

$$= (V_1^{ext} + V_1^{int}) - (V_2^{(e)} + V_2^{int})$$

$$V = V_1 - V_2 \qquad V_1 = V_1^e + V_1^{int}$$

$$V_2 = V_2^e + V_2^{int}$$

The work done can be expressed in term of difference between the K.E in the initial and final configuration

$$W = \sum_i \int_1^2 \bar{F}_i \cdot d\bar{r} = \sum_i \int_1^2 \frac{d}{dt}(m_i v_i) \cdot dr_i$$

$$= \sum_i m_i \int_1^2 \frac{dv_i}{dt} \cdot dr_i$$

$$= \sum_i m_i \int_1^2 \left(\frac{dv_i}{dt} \cdot \frac{dr_i}{dt}\right) dt$$

$$= \sum_i m_i \int_1^2 \left(\frac{dv_i}{dt} \cdot v_i\right) dt$$

$$= \sum_i m_i \int_1^2 \frac{1}{2} d(\bar{v}_i)^2$$

02.02.06

$$= \sum m_i \int_1^2 \frac{1}{2} d(\bar{V}_i)^2 \qquad \text{As} \quad \frac{dV_i}{dt} V_i = \frac{1}{2}\frac{d}{dt}(V_i)^2$$

$$= \sum_i \frac{1}{2} m \bar{V}_i^2 \Big|_1^2$$

$$W = \sum_i T_i \Big|_1^2 = T_2 - T_1 \quad -⑨$$

Com From eq ⑧ + ⑨ we have

$$V_1 - V_2 = T_2 - T_1$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2$$

$$E_1 = E_2$$

*

CONSTAINTS:-

CHAPTER # 8
TAKWALE # 217

DEGREE OF FREEDOM: The number of independent ways in which a mechanical system can move without voilating any constraints which may be imposed on the system is called the number of degree of freedom of that system. In other words the number of degree of freedom is the number of independent variable. A system of N particles frees from the constraints has 3N degree of freedom and if "K" is the number of constraints then degree of freedom will be

degree of freedom = 3N − K

where N is number of particles
and K is number of constraints.

Consider a bead moves in a wire, then number of constraints is y=0, z=0 is two, so the number of particle is one then degree of freedom will be

degree of freedom = 3(1) − 2 = 3 − 2 = 1

16-02-06

CONSTRAINTS:-

The things which restricts the motion of a body is called constraints, when motion of system is restricted we say that constraines are introduced. Normally constraints restricts the motion of the body. The example of constraints are normally a bead sliding on a wire which is restricted to move on a specific path. Similarly gas molecules are constraints in the cylender, they cannot go out from the cylender. A particle placed on the surface of the sphere is constraints to move only on the sphere. A disc rolling on the inclind plane is also the example of the constraints, thus when constraints are introduced into a system the degree of freedom of the system decreases.

A free particle in space have three degree of freedom because no constraints is applied on it.

degree of freedom = 3(1) − 0 = 3

When a particle moving on a plane (Table) then it has only one constraint z=0 and degree of freedom will be degree of freedom = 3(1) − 1 = 2

Here we have relationship of two plane (x,y), z=0

z=0 is called superflous coordinate.

Superflous coordinate is that coordinate which is connected with constraints.

The superflous coordinate is made in the below exemple.

(i) A straigth line on the table.

$$y = a + bx \qquad (z=0 \text{ Superflous coordinate})$$

(ii) In case of parabola

$$y = bx^2 \qquad (z=0)$$

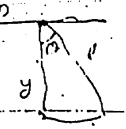
x & y are independent coordinate and z is the superflous coordinate in this case.

16-02-06

When constraints are introduced the number of independent coordinate reduces this dependence of a problem on coordinate is not accepted in science. In order to avoid this dependence the generalize coordinate are introduced.

In case of smple pendulum x, y is independent coordinate and $z=0$ is superflous coordinate.

The degree of freedom of simple pendulum is degree of freedom $= 3N-P$



$= 3(1)-(1) = 3-1$

$= 2$

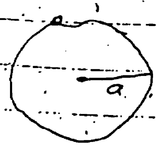
(ii) Any body in a plane has two degree of freedom. restriction or constraints $= 1$

degree of freedom $= 3(1)-1 = 2$

(iii) A particle on the surface of sphere has two degree of freedom. No. of constraints is one because $a =$ constant, the body cannot enters the sphere.

degree of freedom $= 3N-K = 3(1)-1$

$= 2$



In case of rigid body

$|\vec{r}_i - \vec{r}_j| =$ constant $= |\vec{r}_{ij}|$

Constraints may be classified into the following types.

① **Holonomic Constraints:-** A holonomic constraints is that type of constraints which may be expresse in the form of the equation relating the coordinate of the system and time. The general form of such equations, for a system of N-particles

$F_i(x_1, y_1, z_1, x_2, y_2, z_2 \ldots\ldots x_n, y_n, z_n, t) = 0$

For a single particle we have

$F(x, y, z, t) = 0 \qquad i = 1, 2, 3 \ldots r$

where F_i is some function of coordinate

16-02-06

The simplest exemple of the holonomic (coordinate constraints) is the rigid body, where the constraints are expressed by equation of the form

$(r_i - r_j)^2 - c_{ij}^2 = 0$

A particle constraints to move along a curve or on a given surface is another exemple of the holonomic constraints.

② **Non Holonomic Constraints:-**

Non Holonomic constraints are not expressed in the form in which the holonomic constraints are expressed. It may be in the form of inequality as such that

$F_i(x_1, y_1, z_1, x_2, y_2, z_2 \ldots\ldots x_N, y_N, z_N, t) \geq 0$ or $<$

These constraints involve the exemple of particle placed on the surface of the sphere is also nonholonomic, for it can be expressed as an inequality $r^2 - a^2 \geqq 0$

where 'a' is the radius of the sphere. Thus in gravitational field a particle placed on the top of the sphere will slide down the surface part of the way but will eventualy fall off.

③ **Scleronomous Constraints:-**

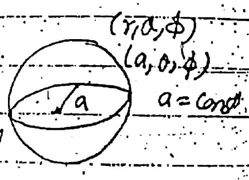
Scheronomous constraints are those constraints ~~are~~ which are independent of the time, means that they does not depends upon time, for exemple a rigid body $|\vec{r}_i - \vec{r}_j| = r_{ij}$

The rigid body is independent of time.

A bead sliding on a rigid curved wire fixed in space is obviously subject to Scheronomous constraints.

④ **Rheonomous Constraints:-** Rheonomous Constraints are those Constraints which explicitly (totaly) dependent on time. For example a pendulum whose length is changing $l = l(t)$.
A ballon whose radius is changing with time Then it is Rheonomous Constraints. A sphere which have radius 'a' and which is changing w.r.t. time $a = a(t)$
where 'a' is the radius of the sphere which is totaly dependent on time.
In case of Pendulum when $z=0$ and length is changing Then $x^2+y^2=l^2$, $z=0$
where $l=$ const Then it is a function of θ only

(1) Consider a bead on the surface of the sphere. As far as the bead is in Contact with the spherical surface Then $x^2+y^2+z^2=a^2$.



It is Holonomic Constraints
When bead is not in contact with the spherical surface Then $x^2+y^2+z^2>a^2$
Then it is non holonomic Constraint. The bead Cannot goes inside the surface of the sphere. It is Constraint on the bead.

* **HOME ASSIGNMENT**

PROBLEM motion of Particle subjected to resistive force. Solve $F = -kmv$
Find the velocity and position 'x'.

SOLUTION we know That

$$F = -kmv \qquad \text{And } F = ma = m\frac{dv}{dt}$$

$$m\frac{dv}{dt} = -kmv$$

$$\frac{dv}{dt} = -kv$$

$$\Rightarrow dv = -k\,v\,dt$$

$$\text{or} \quad \frac{dv}{v} = -k\,dt$$

Integrating both sides we have

$$\int \frac{dv}{v} = -\int k\,dt$$

$$\ln v = -kt + c \quad —Ⓐ$$

Applying initial condition when $t=0$ Then $v=0$
Then putt in eq Ⓐ we get

$$\ln(0) = -k(0) + c$$

$$\Rightarrow c = 0$$

putting value of "c" in eq Ⓐ we have

$$\ln v = -kt + 0$$

$$\ln v = -kt$$

Taking antilog on both sides we have

$$v = e^{-kt} \qquad \Rightarrow \boxed{v = e^{-kt}}$$

As $v = \frac{dx}{dt}$ Then

$$\frac{dx}{dt} = e^{-kt}$$

$$\Rightarrow dx = e^{-kt}\,dt$$

Integrating both sides we have

$$\int dx = \int e^{-kt}\,dt$$

$$x = -\frac{e^{-kt}}{k} + d \quad — Ⓑ$$

Applying initial condition when $t=0$ Then $x=0$

$$0 = -\frac{e^0}{k} + d \Rightarrow d = \frac{1}{k}$$

putt in eq Ⓑ we get

$$x = -\frac{e^{-kt}}{k} + \frac{1}{k}$$

$$\Rightarrow \boxed{x = \frac{1-e^{-kt}}{k}}$$

17.02.06

PRINCIPLE OF VIRTUAL WORK :-

Virtual displacement is the infinitismal displacement of a system refers to a change in the configuration of a system. The displacement is called virtual to distingh it from the actual displacement which occur in time interval dt, during which the forces of constraints may be changing.

Suppose a system is in equilibrium, the total force on each particle is ($F_i = 0$) is zero, then clearly the virtual work $F_i \cdot \delta r_i = 0$ will also be zero and δr_i is the virtual displacement. Then for all particles the sum will zero.

$$\sum_i F_i \cdot \delta r_i = 0$$

We decompose F_i into two forces $F_i^{(a)}$ applied forces and $f_i^{(c)}$ forces of constraints. $F_i = F_i^{(a)} + f_i^{(c)}$

Then $$\sum F_i^{(a)} \cdot \delta r_i + \sum f_i^{(c)} \cdot \delta r_i = 0$$

We know that "virtual work due to forces of constraints is zero it holds for rigid body. If a body moves on a surface the forces of constraints is ⊥ to the surface while virtual displacement tangent to it $\theta = 90°$"

Then $$\sum F_i^{(a)} \cdot \delta r_i = 0.$$

This is called principle of virtual work, which states that

"Virtual work done by the applied forces acting on a system in equilibrium is zero, provided that no frictional forces are present."

δx infinitismal small change in 'x' or virtual change in 'x'. Physically it is almost zero. Therefore it is called virtual change.

Definition of virtual work is

$$\delta w = F_i \cdot \delta \vec{r_i}$$

where δr_i is the virtual displacement.

33

17-02-06

D'ALEMBERT'S PRINCIPLE :-

In the principle of virtual work $\sum F_i^{(a)} \cdot \delta r_i = 0$, the coefficient δr_i can no longer ber zero and $F_i^{(a)} \neq 0$. Since δr_i is not independent but connected to constraints.

To obtain this principle, the equation of motion will

$$F_i = \dot{P}_i$$
$$F_i - \dot{P}_i = 0$$

or $$\sum_i (F_i - \dot{P}_i) \delta r_i = 0$$

This is mathamatical form of D'Alembert principle.

Constraints are unknown and it is very difficult to find the values of constraints force. The virtual work is given as $\delta w_i = \bar{F}_i \cdot \delta r_i$

D'Alembert used the principle of virtual work.

Lagrang'es used D'Alembert principle to develop a new mechanics called as "Lagrang's Mechanics".

Mathamatically the D. Alembert principle is

$$\sum (\bar{F}_i - \dot{\bar{P}}_i) \cdot \delta r_i = 0$$

Consider a system of n generalized coordinate q_j ($j = 1, 2, 3 \ldots n$)

Suppose the system undergoes a certain displacement in configuration space in such a way that it does not take any time and is consistent with constraints on the system.

Such displacement is called virtual displacement because they do not represent the actual displacement. (Non actual displacement)

The work done by forces of constraints is zero. The forces of constraints is at right angle to the displacement.

Let the virtual displacement of ith particle of a given system is represented by δr_i

If the system is in equilibrium the resultant force on the ith particle must be zero $\vec{F}_i = 0$

then $\vec{F}_{(i)} \cdot \delta r_i = 0$

and for entire system

$$dw = \sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0 \quad \text{—(1)}$$

The resultant force $\vec{F}_i$ is made of two forces (not considering the friction)

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{F}_i^{(c)}$$

Applied force and forces of constraints

then eq(1) will be

$$dw = \sum_i (F_i^{(a)} + F_i^{(c)}) \cdot \delta r_i$$

$$dw = \sum_i F_i^{(a)} \cdot \delta r_i + \sum_i F_i^{(c)} \cdot \delta r_i \quad \text{—②}$$

As the virtual work by the forces of constraints is zero $\sum_i F_i^{(c)} \cdot \delta r_i = 0$ — ③

Since constraints are perpendicular to $\delta \vec{r}_i$ which is virtual displacement

then

$$\sum_i F_i^{(a)} \cdot \delta \vec{r}_i = 0 \quad \text{—(4)}$$

Equation ③ will not hold when friction is there, because the friction act opposite to the displacement.

Most of the system we come across in mechanics are not in static equilibrium. The principle of eq④ must be modified to include the dynamic system as well.

we have

$$\vec{F}_i = \dot{\vec{P}}_i$$

or

$$\vec{F}_i - \dot{\vec{P}}_i = 0 \quad \text{—⑤}$$

Equation ⑤ states that the dynamic system appear to be in equilibrium under the action of applied force $\vec{F}_i$

and an equal and opposite "effective force" $\dot{\vec{P}}_i$

$$\left(\because P = mv \quad \dot{\vec{P}} = ma = F\right)$$

The $\dot{\vec{P}}_i$ is also called kinetic reaction

The principle of virtual work can be putt in a useful way

$$\sum_i (\vec{F}_i - \dot{\vec{P}}_i) \cdot \delta r_i = 0 \quad \text{—⑥}$$

In eq⑥ all the forces are applied forces and forces of constraints do not appear in consideration and $(-\dot{\vec{P}}_i)$ is called as reverse effective force which we call the name of kinetic reaction. The eq⑥ is the mathematical form of D'Alembert Principle.

* *

In Generalized coordinate

$$\vec{r}_i = \vec{r}_i(q_j's) = r_i(q_1, q_2, q_3, \ldots q_j)$$

virtual displacement will be given as

$$\text{virtual displacement} = \delta \vec{r}_i = \sum_j \frac{\partial r_i}{\partial q_j} \delta q_j \quad \text{—①}$$

The velocity of the ith particle will be

$$\vec{r}_i = \vec{r}_i(q_j)$$

$$\left(V_i = \frac{d\vec{r}_i}{dt} = \frac{d}{dt} \vec{r}_i(q_j) = \sum_j \frac{\partial r}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$V_i = \dot{\vec{r}}_i = \sum_j \left(\frac{\partial r_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial \vec{r}_i}{\partial t}\right) \quad \text{—②}$$

The work done in this case will be

$$\delta w = \vec{F}_i \cdot \delta \vec{r}_i$$

Then

$$\delta w = \sum_i \left(\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) \delta q_j \quad \text{—③}$$

where

$$\vec{F}_i \cdot \frac{\partial r_i}{\partial q_j} = Q_j$$

where Q_j is called Generalized force

Then eq③ will be given as

$$\delta W = \sum_j Q_j \cdot \delta q_j$$

Qj may not have dimension of force but $Q_j\, \delta q_j$ must have dimension of work

LAGRANGE'S EQUATION FROM D'ALEMBERT'S PRINCIPLE :-

Notes

we have Two difficulties for solving our Problems one is The forces of constraints which are unknown and we donot consider These forces, The second one is dependence on the coordinate for This purpose we introduce the generalize coordinates. Lagrange's Equation is mathamatical expression for expressing Newton 2nd law of motion in Term of generalize coordinates Thus the generalization of Newton 2nd law of motion is called Lagrange's Equation we have

$$\bar{r}_i = r_i(q_j, t)$$

$$\Rightarrow \delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\, \delta q_j$$

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \dots\dots \frac{\partial \bar{r}_i}{\partial q_N}\delta q_N \quad \text{—Ⓐ}$$

As we have

$$\bar{r}_i = \bar{r}_i(q_j, t)$$

$$\dot{\bar{r}}_i = V = \frac{d}{dt}\bar{r}_i = \sum_j \frac{\partial r_i}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial \bar{r}_i}{\partial t}$$

$$\Rightarrow \dot{\bar{r}}_i = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \quad \text{—Ⓑ} \qquad \left(\dot{q}_j = \frac{\partial q_j}{\partial t}\right)$$

The work will be gevin as

$$\delta W = F_i \cdot \delta r_i$$

putting the values

17-02-06

$$\delta W = F_i \cdot \delta r_i = F_i \cdot \frac{\partial r_i}{\partial q_j}\, \delta q_j$$

$$\delta W = \left(F_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right)\delta q_j$$

$$F_i \cdot \delta r_i = \delta W = \sum_j Q_j\, \delta q_j \quad \text{—Ⓒ} \qquad \text{As } Q_j = \sum_j \left(F_i \cdot \frac{\partial r_i}{\partial q_j}\right)$$

where Qj is called generalized force which may not have The dimension of force.

But $Q_j \delta q_j$ must have dimension of work.

From D'Alembert principle we have

$$\sum (F_i - \dot{P}_i) \cdot \delta r_i = 0$$

$$\sum F_i \cdot \delta r_i - \sum \dot{\vec{P}}_i \cdot \delta r_i = 0 \quad \text{—①}$$

In eq ① the first Term is present in eq Ⓒ we consider The 2nd Term

$$\sum_i \dot{\vec{P}} \cdot \delta r_i = \sum_i \frac{d}{dt}(m_i \dot{\vec{r}}) \cdot \delta \bar{r}_i$$

$$\text{or} \quad \sum_i \dot{\vec{P}} \cdot \delta r_i = \sum_i m_i \ddot{\vec{r}} \cdot \delta r_i$$

putting The value of δr_i from eq Ⓐ

$$\sum_i \dot{\vec{P}} \cdot \delta r_i = \sum_i m_i \ddot{\vec{r}} \cdot \frac{\partial r_i}{\partial q_j}\delta q_j \quad \text{—②}$$

In order to write eq ② in alternate way

Applying product rule we have

$$\sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}} \cdot \frac{\partial r_i}{\partial q_j}\right) = \sum_i m_i \ddot{\vec{r}}_i \frac{\partial r_i}{\partial q_j} + \sum_i m_i \dot{\vec{r}} \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$$

$$\text{or} \quad \sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - \sum_i m_i \dot{\vec{r}} \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \quad ③$$

From eq Ⓑ we have

$$V = \dot{\vec{r}} = \frac{\partial r_i}{\partial t} = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}$$

Diff w.r.t to ∂q_j

$$\frac{d}{dq_j}\left(\frac{\partial \vec{r}_i}{\partial t}\right) = \sum_k \frac{\partial^2 r_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t \cdot \partial q_j}$$

38 17.02.06

Here in above we can change order of differentiation $\dot{q}_j$ with respect to t we have

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 r_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 r_i}{\partial q_j \partial t} = \frac{\partial \dot{\vec{r}}_i}{\partial Q_j} \quad \text{—④}$$

$\dot{q}_j$ is called generalized velocity which may not have dimension of velocity. In eq ④ the order of differentiation w.r.t q_j and t can be interchanged.

If we differentiate eq ③ with respect to $\dot{q}_j$ we get

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \quad \text{—⑤}$$

Substitute the value of eq ⑤ & eq ④ in eq ③ we have

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i \dot{\vec{r}} \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right]$$

$$= \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right] \quad \text{—ⓢⓐ}$$

we consider

$$\frac{\partial}{\partial q_j}(m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) = m_i \dot{r}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j} + m_i \frac{\partial}{\partial q_j}\dot{r} \cdot \dot{\vec{r}}$$

$$= (m_i \dot{\vec{r}} + m_i \dot{\vec{r}})\frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

or $$\frac{\partial}{\partial q_j}(m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) = 2 m_i \dot{\vec{r}} \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

Dividing by ② we have

$$\frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \dot{\vec{r}}_i^2\right) = m_i \dot{\vec{r}} \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

putting this value in eq ⑤ⓐ we get

$$\sum_i \left(m_i \ddot{\vec{r}} \cdot \frac{\partial r_i}{\partial q_j}\right) = \sum_i \left[\frac{d}{dt}\sum_i \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m_i \dot{r}_i^2\right) - \sum_i \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \dot{\vec{r}}^2\right)\right]$$

39 17.02.06

or $$\sum \vec{P} \cdot \delta r_i = \sum_i \left(m_i \ddot{\vec{r}} \cdot \frac{\partial r_i}{\partial q_j}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T) \quad \text{—⑥}$$

where $T = \frac{1}{2} m_i \dot{\vec{r}}^2$ is the K.E of the system

we have D'Alembert principle

$$\sum (F_i - \vec{P}_i) \cdot \delta r_i = 0$$

$$\sum F_i \cdot \delta r_i - \sum \vec{P}_i \cdot \delta r_i = 0$$

putting the values from ⓒ & eq ⑥ we have

$$\sum Q_j \delta q_j - \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T)\right] \cdot \delta r_i = 0$$

or $$\sum \left[Q_j - \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T)\right]\right]\delta q_j = 0$$

∴ The virtual displacement δq_j are all independent from each other. The above eq is true only when coefficient of each displacement is zero

$$Q_j - \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T)\right] = 0$$

or $$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}T = Q_j \quad \text{—⑦}$$

where Q_j is the generalized force

Eq ⑦ is valid for ~~conservative as well as~~ for non conservative force. For conservative we have ↓

Conservative force :- If the system is conservative it means that forces are derivable from scalar potential, then $\vec{F} = -\nabla_i V$

As $Q_j = \sum_j F_i \cdot \frac{\partial r_i}{\partial q_j}$ then

$$Q_j = F_{ix}\frac{\partial x_i}{\partial q_j} + F_{iy}\frac{\partial y_i}{\partial q_j} + F_{iz}\frac{\partial z_i}{\partial q_j}$$

In place of $\vec{r}_i$ we are writing x_i, y_i, z_i

17.02.06

$$Q_j = \sum -\frac{\partial V_i}{\partial x_i}\frac{\partial x_i}{\partial q_j} - \frac{\partial V_i}{\partial y_j}\frac{\partial y_i}{\partial q_j} - \frac{\partial V_i}{\partial z_i}\frac{\partial z_i}{\partial q_j}$$

$$Q_j = -\sum_j \frac{\partial V_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad -⑧$$

V is a Scalar function $V = P.E$

More ever $V = V(q_j, s)$ Then $\frac{\partial V}{\partial q_j} = 0$

Comparing eq ⑦ and eq ⑧ we have

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}(T) = -\frac{\partial V}{\partial q_j}$$

$$\text{or} \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\text{or} \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \left(\frac{\partial (T-V)}{\partial q_j}\right) = 0$$

$$\text{or} \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}(T-V) = 0 \quad -⑨$$

~~whe~~ By adding $\frac{\partial V}{\partial \dot{q}_j}$ to above equation we have

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial \vec{q}_j}(T-V) = 0$$

where $T - V = L$ Then above eq ⇒

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad -⑩}$$

This is The Equation of motion for Lagrangian where $L = T - V$ and is called Lagrangian. It is equal to The difference of K.E & P.E of the Syste

$L = L(q_j, \dot{q}_j, t)$

which is a function of Position, velocity and Time but T is a function of velocity and P.E is function of position

*

*

17.02.06

SIGNIFICANCE OF LAGRANGIAN FORMULATION /EQUATION :-

The significance or importance of the Lagrangian formulation is discussed on the following points

① The Lagrangian formulation is not a new development, but its is an equivalent way of solving different problems.

② Both Lagrangian and Newton Equations are second order differential Equation, when solved They would describe motion of system of Particle They are derived from different Consideration.

③ Newtonian formulation Consider forces i-e Applied force acting on a system, this is difficult when certain constraints are acting on The system (because forces of constraints are one of The unknown of The Problems)

④ In Lagrangian Formulation The Consideration is Kinetic energy and Potential Energy of a System Consideration of force does not enter into this Consideration, Both Potential and Kinetic Energies are Scalar quantities, Both remain invarient under coordinate Transformation. But in Newtonian The force is most important Things because accelration is produced due to force.

The generalize coordinate may be selected in an independent way, so That The problem is made simpler.

If constraints are present The Lagrangian formulation is useful to apply to a Problem.

*

*

EQUIVALENCE OF NEWTONIAN & LAGRANGIANS

OR Derivation of Newton 2nd law from Lagran

The Lagrange's Equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad — ①$$

where $L = T - V$ it is derived on the basis of the D'Alembert' Principle.

Let us consider eq ① in rectangular Coordinates x_j Then $q_j = x_j$ is the Coordinate $(j = 1,2,3 \ldots)$

Then $x_j = (x_1, x_2, x_3) \ldots)$

the y, z does not comes in it because here x_1 is for 'x', x_2 is for 'y' and x_3 is for 'z' in the Cartesian Coordinates.

$$x_j = (\underset{\overset{\downarrow}{x}}{x_1}, \underset{y}{x_2}, \underset{z}{x_3})$$

Putting the value of q_j in eq ① we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}_j} - \frac{\partial L}{\partial x_j} = 0$$

or $$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{\vec{x}}_j} - \frac{\partial (T-V)}{\partial x_j} = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{x}}_j} - \frac{d}{dt}\frac{\partial}{\partial \dot{\vec{x}}_j}V - \frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = 0$$

As Kinetic Energy is a function of $\dot{x}_j$ (V) and Potential Energy is a function of position x_j. Then

$$\frac{\partial V}{\partial \dot{\vec{x}}_j} = 0 \quad \& \quad \frac{\partial T}{\partial x_j} = 0 \text{ putt in above}$$

we left $$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{x}}_j} + \frac{\partial V}{\partial \vec{x}_j} = 0$$

or $$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{x}}_j} = -\frac{\partial V}{\partial \vec{x}_j} \quad — ②$$

We know that $T = K.E = \frac{1}{2}m\dot{x}_j^2$ Then

$$T = \frac{1}{2}m\dot{x}_j^2$$

Then $$\frac{\partial T}{\partial \dot{x}_j} = \frac{2}{2}m\dot{x}_j = m\dot{x}_j$$

& $$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{x}}_j} = m\frac{d}{dt}\dot{\vec{x}}_j = m\ddot{\vec{x}}_j = m\vec{a} = \vec{P}$$

or $$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{x}}_j} = \dot{\vec{P}} \quad — ③$$

For Potential Energy for Conservative system

$$\vec{F} = -\vec{\nabla}V$$

or $$\vec{F} = -\frac{\partial V}{\partial x_j} \quad — ④$$

Then putting the values of eq ④ & ③ in eq ②

we get $$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{x}}_j} = -\frac{\partial V}{\partial x_j}$$

$$\boxed{\dot{\vec{P}} = \vec{F}} \quad (\text{As } \vec{P} = m\vec{v} \text{ And } \dot{\vec{p}} = ma)$$

$$\Rightarrow ma = F$$

$$\Rightarrow \boxed{\vec{F} = ma}$$

which is Second law of motion is verified

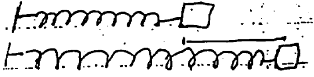
★ ★

PROBLEM:- Write the Equation for the mass attached to a spring by Newtonian as well as by the Lagrange's formulation:-

SOLUTION:-

Newtonian Formulation:

When force is applied on the body then it will displace from its original position.



Then Applying Newton 2nd law of motion

$$F = ma$$

17.02.06

$F = -kx$

By Hook's law we have

$F = ma$

then $-kx = ma$

or $ma + kx = 0$ (As $a = \ddot{x}$)

$m\ddot{x} + kx = 0$

or ~~$\ddot{x} + (\frac{k}{m})x = 0$~~

or $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{k/m}$

$\ddot{x} + \omega^2 x = 0$

It is second order differential Equation

LAGRANGIAN FORMULATION:-

In LAGRANGIAN we have

$L = T - V$ $T = \frac{1}{2}m\dot{x}^2$

$W = F_{av}\, x$

$= (\frac{0 + kx}{2})x = \frac{1}{2}kx^2$

$W = \frac{1}{2}kx^2 \Rightarrow V = \frac{1}{2}kx^2$

& Then Lagrangain will be $T = \frac{1}{2}m\dot{x}^2$

$L = T - V$

$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ — ①

Diff w.r.t $\dot{x}$ we have

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$

& diff w.r.t x we have

$\frac{\partial L}{\partial x} = -kx$

Then putting these values in Lagrangain we have

$\frac{d}{dt}\cdot\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

$m\ddot{x} - (-kx) = 0$

Both gives the same result

$m\ddot{x} + kx = 0$

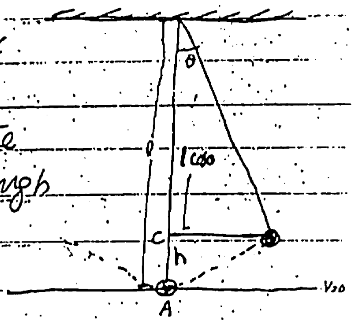
$\ddot{x} + \frac{k}{m}x = 0$ where

$\ddot{x} + \omega^2 x = 0$ $\omega = \sqrt{\frac{k}{m}}$

17.02.06

SIMPLE PENDULUM:- ①

write the Equation of motion for simple pendulum.



we have to write the Equation of motion for simple pendulum. The "l" is the total length of the wire and 'θ' is the angle through which the pendulum moves

The kinetic Energy of the bob is

$T = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2)$

when l = length = constant then $\dot{l} = 0$ & $\dot{l}^2 = 0$

then $T = \frac{1}{2}m(0 + l^2\dot{\theta}^2) = \frac{1}{2}ml^2\dot{\theta}^2$

Let say OA is its length, For Potential Energy the reference is $V = 0$ is l below the point of suspension

$V = mgh = mg(OA - OC)$

$V = mg(l - l\cos\theta)$ $OA = l$

As we know that the Lagrangain is the difference between potential Energy and kinetic Energy $OC = l\cos\theta$

$L = T - V$

$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$ — ①

∴ The motion of simple pendulum can be described by one coordinate $(\theta = q_1)$ In Lagrangain Equation q will be replaced θ we get

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$ — Ⓐ

From eq ① diff it w.r.t $\dot{\theta}$ we have

②

$$\frac{\partial L}{\partial \dot\theta} = \frac{1}{2}ml^2(2\dot\theta) = ml^2\dot\theta$$

or $\frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta$

& $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = ml^2\ddot\theta$

Similarly diff eq (i) w.r.t θ we have

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

we have $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0$

putting the values we have

$ml^2\ddot\theta + mgl\sin\theta = 0$

Dividing by ml we have

$l\ddot\theta + g\sin\theta = 0$

In case of simple pendulum $\sin\theta \simeq \theta$ Then

$l\ddot\theta + g\theta = 0$

Dividing by l we have

$\ddot\theta + \frac{g}{l}\theta = 0$

or $\ddot\theta + \omega^2\theta = 0$ where $\omega = \sqrt{g/l}$

This is the Equation of simple harmonic motion in case of simple pendulum. In case of simple pendulum the only constraints is that the length l is constant

* *

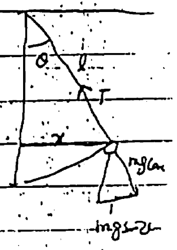
MATI ULLAH 0345-9798233
M.SC (Prevous)
University of Peshawar.
FRIDAY 8:43 17 Feberuary 2006.

---

③ 12.02.06

If we solve **NEWTONIAN WAY**:-

The motion of pendulum by Newtonian we have



$T - mg\cos\theta = 0$

$T = mg\cos\theta$ balance each other

The bob moves along the direction of $mg\sin\theta$ $F = ma = -mg\sin\theta$

$\Rightarrow a = -g\sin\theta$

& $x = l\theta \Rightarrow \dot x = l\dot\theta \Rightarrow (\ddot x = l\ddot\theta)$

As $\ddot x = a$ Then

$l\ddot\theta = a = \ddot x = -g\sin\theta$

$\Rightarrow l\ddot\theta + g\sin\theta = 0$

In case of S. pendulum $\sin\theta \simeq \theta$ Then

$l\ddot\theta + g\theta = 0$

or $\ddot\theta + \frac{g}{l}\theta = 0$ $\omega = \sqrt{g/l}$

or $\ddot\theta + \omega^2\theta = 0$

In this case we have ignored here friction, & constraints forces we have consider here only External forces.

* *

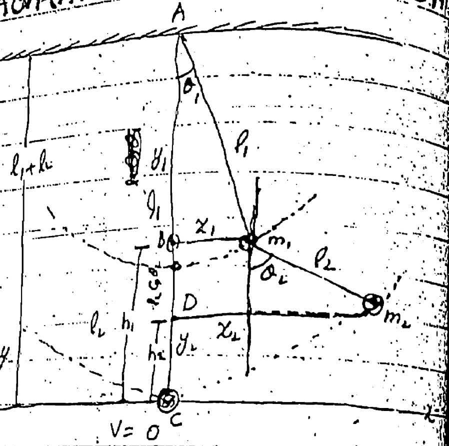
MATI ULLAH 0345-9798233
* M.SC (Prevous)
Surrani Bannu
Physics Deptt
University of Peshawar.

17.02.06

[illegible]

Double PENDULUM:-

Write The Equation of MOTION FOR DOUBLE PENDULUM USING LAGRANGAIN FORMULATION



The Lagrangian Equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = 0$$

Where $L = T - V$

Now we have to find T (Kinetic Energy) and (V) potential energy of the Double pendulum.

The K.E is given as

$$T = \frac{1}{2} m_1 (\dot x_1^2 + \dot y_1^2) + \frac{1}{2} m_2 (\dot x_2^2 + \dot y_2^2) \quad — Ⓐ$$

we have

$x_1 = l_1 \sin\theta_1$, $y_1 = l_1 \cos\theta_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ $y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$

Then

$\dot x_1 = l_1 \cos\theta_1 \dot\theta_1$ & $\dot y_1 = -l_1 \sin\theta_1 \dot\theta_1$

$\dot x_2 = l_1 \cos\theta_1 \dot\theta_1 + l_2 \cos\theta_2 \dot\theta_2$ $\dot y_2 = -l_1 \sin\theta_1 \dot\theta_1 - l_2 \sin\theta_2 \dot\theta_2$

putting these values in eq Ⓐ we have

$$T = \frac{1}{2} m_1 \left((l_1 \cos\theta_1 \dot\theta_1)^2 + (-l_1 \sin\theta_1 \dot\theta)^2 \right) + \frac{1}{2} m_2 \left((l_1 \cos\theta_1 \dot\theta_1 + l_2 \cos\theta_2 \dot\theta_2)^2 + (-l_1 \sin\theta_1 \dot\theta_1 - l_2 \sin\theta_2 \dot\theta_2)^2 \right]$$

$$= \frac{1}{2} m_1 \left(l_1^2 \dot\theta_1^2 (\cos^2\theta_1 + \sin^2\theta_1) \right)$$
$$+ \frac{1}{2} m_2 \left[l_1^2 \cos^2\theta_1 \dot\theta_1^2 + l_2^2 \cos^2\theta_2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos\theta_1 \cos\theta_2 \right.$$
$$\left. l_1^2 \sin^2\theta_1 \dot\theta_1^2 + l_2^2 \sin^2\theta_2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin\theta_1 \sin\theta_2 \right]$$

$$T = \frac{1}{2} m_1 (l_1^2 \dot\theta_1^2) + \frac{1}{2} m_2 \left(l_1^2 \dot\theta_1^2 (\cos^2\theta_1 + \sin^2\theta_1) + l_2^2 \dot\theta_2^2 (\cos^2\theta_1 + \sin^2\theta_2) + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \right.$$

49

17.02.06

$$T = \frac{1}{2} m_1 (l_1^2 \dot\theta_1^2) + \frac{1}{2} m_2 (l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)$$

or

$$T = \frac{1}{2} m_1 (l_1^2 \dot\theta_1^2) + \frac{1}{2} m_2 (l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)) \quad — Ⓐ$$

For Potential Energy reference is Taken below The point of suspension

$$V = m_1 g h_1 + m_2 g h_2$$

Here $h_1 = AC - AB$
$= (l_1 + l_2) - l_1 \cos\theta_1$

$h_2 = AC - AD = (l_1 + l_2) - (l_1 \cos\theta_1 + l_2 \cos\theta_2)$

Then Potential Energy will be

$$V = m_1 g [(l_1 + l_2) - l_1 \cos\theta_1)] + m_2 g [(l_1 + l_2) - (l_1 \cos\theta_1 + l_2 \cos\theta_2)]$$

Then The Lagrangian will be

$L = T - V$

$$L = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left[l_1 \dot\theta_1 + l_2 \dot\theta_2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) \right]$$
$$- m_1 g ((l_1 + l_2) - l_1 \cos\theta_1) - m_2 g ((l_1 + l_2) - (l_1 \cos\theta_1 + l_2 \cos\theta_2)) \quad — Ⓑ$$

The Lagrangian Equation for θ_1 is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} = 0 \quad — ①$$

The Lagrangian Equation for θ_2 is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} - \frac{\partial L}{\partial \theta_2} = 0 \quad — ②$$

Differenting eq Ⓑ w.r.t $\dot\theta_1$ we have

$$\frac{\partial L}{\partial \dot\theta_1} = m_1 l_1^2 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1 - \theta_2)$$

&

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = m_1 l_1^2 \ddot\theta_1 + m_2 l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1 - \theta_2) (\dot\theta_1 - \dot\theta_2)$$

Diff eq Ⓑ w.r.t θ_1 we have

$$\frac{\partial L}{\partial \theta_1} = - m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1$$

putting these values in eq. ① we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} = 0$$

$$m_1 l_1^2\ddot\theta_1 + m_2 l_1^2\ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2) + m_2 l_1 l_2 \dot\theta_1\dot\theta_2 \sin(\theta_1-\theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0$$

or

$$m_1 l_1^2\ddot\theta_1 + m_2 l_1^2\ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) - \cancel{m_2 l_1 l_2 \dot\theta_1\dot\theta_2 \sin(\theta_1-\theta_2)} + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + \cancel{m_2 l_1 l_2 \dot\theta_1\dot\theta_2 \sin(\theta_1-\theta_2)} + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0$$

or

$$m_1 l_1^2\ddot\theta_1 + m_2 l_1^2\ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0 \quad ③$$

✓

Now consider $q_j = \theta_2$ then we have diff eq Ⓑ w.r.t $\dot\theta_2$ we have

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1-\theta_2)$$

Now

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)$$

& diff w.r.t θ_2 eq. Ⓑ we have

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)(-1) - m_2 g l_2 \sin\theta_2$$

$$= m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2$$

we know that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} - \frac{\partial L}{\partial \theta_2} = 0$$

putting the values we have

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1-\theta_1)(\dot\theta_1-\dot\theta_2) - m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

or

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) + \cancel{m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)} - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) - \cancel{m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)} + m_2 g l_2 \sin\theta_2 = 0$$

17.02.06

or

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 = 0 \quad — ④$$

Equation ③ & ④ gives the motion of pendulum we make the substitution in eq ③ & ④ such that

$m_1 = m_2 = m$ & $l_1 = l_2 = l$ $\theta_1 - \theta_2 = \theta - \theta = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2 \ddot\theta_1 + m_2 l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0$$

similarly $\sin\theta \approx \theta$ & $\cos\theta = 1$ $\theta_1 - \theta_2 = \theta - \theta = 0$

$$\Rightarrow m l^2 \ddot\theta_1 + m l^2 \ddot\theta_1 + m l^2 \ddot\theta_2 + m g l \theta_1 + m g l \theta_1 = 0$$

$$2 l^2 \ddot\theta_1 + m l^2 \ddot\theta_2 + 2 m g l \theta_1 = 0$$

or $$2 l^2 \ddot\theta_1 + m l^2 \ddot\theta_2 = -2 m g l \theta_1$$

Dividing by m we have

$$2 l^2 \ddot\theta_1 + \cancel{m} l \ddot\theta_2 = -2 g l \theta_1$$ Dividing by 'l'

$$2 l \ddot\theta_1 + l \ddot\theta_2 = -2 g \theta_1$$

$$\Rightarrow \boxed{2 l \ddot\theta_1 + l \ddot\theta_2 = -2 g \theta_1} \quad — Ⓐ$$

Similarly eq ④ ⇒

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} - \frac{\partial L}{\partial \theta_2} = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

make substitution $l_1 = l_2 = l$, $m_1 = m_2 = m$, $\sin\theta \approx \theta$

$$\Rightarrow m l^2 \ddot\theta_2 + m l^2 \ddot\theta_1 (1) - m_2 l^2 \dot\theta_1^2 (0) + m g l \theta_2 = 0$$

$$m l^2 \ddot\theta_2 + m l^2 \ddot\theta_1 + m g l \theta_2 = 0$$

Dividing by ml we have

$$l \ddot\theta_2 + l \ddot\theta_1 + g \theta_2 = 0$$

or $$\boxed{l \ddot\theta_2 + l \ddot\theta_1 = -g \theta_2} \quad — Ⓑ$$

Thus the eq Ⓐ & eq Ⓑ are the equation of motion for double pendulum.

* * * * *

18-02-06

GENERAL EXPRESSION FOR KINETIC ENERGY OR SHOW THAT $T = \frac{1}{2}\sum_{\ell} \dot{q}_\ell P_\ell$

In derivation of Lagrangies Equation the kinetic Energy of the system was expressed in Term of Cartesian (Coordinates) component of velocity. Thus

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i V_i^2 \quad \text{—} ①$$

Now we wish to express the kinetic Energy in Term of generalize coordinates q_j $(j = 1, 2, 3 \ldots n)$ using Lagrange's Equation.

The position of the ith Particle will be

$$\vec{r}_i = \vec{r}_i(q_j, t)$$

The velocity will be

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \quad \text{—} Ⓐ$$

Squaring the above Term but only changing the indixes of one Term

$$\dot{\vec{r}}_i^{\,2} = \dot{\vec{r}}_i \dot{\vec{r}}_i = \sum_{j,k} \left(\frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t}\right)\left(\sum_k \left(\frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t}\right)\right)$$

$$\dot{\vec{r}}_i^{\,2} = \sum_{j,k} \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k} \dot{q}_j \dot{q}_k + \left(\frac{\partial r_i}{\partial t}\right)^2 + 2\sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j \frac{\partial r_i}{\partial t}$$

√ putting this value in eq ① we have

$$T = \sum_i \frac{1}{2} m_i \left(\sum_{j,k} \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k} \dot{q}_j \dot{q}_k + \left(\frac{\partial r_i}{\partial t}\right)^2 + 2\sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j \frac{\partial r_i}{\partial t}\right)$$

or

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c \quad \text{—} ②$$

where

$$\left.\begin{aligned} a_{jk} &= \sum_{j,k} \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k} \\ b_j &= \sum_i m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial t} \\ c &= \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t}\right)^2 \end{aligned}\right\} \quad \text{—} ③$$

18-02-06

When the Transformation Equation Ⓐ is independent of time then the Term b_j and c will be equal to zero because T is independent of Time.

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k = \sum_{j,\ell} a_{j\ell} \dot{q}_j \dot{q}_\ell \quad \text{—} ④$$

Differ this above Term is Time independent thing, then a_{jk} is constant and we have two variable $\dot{q}_j$ $\dot{q}_k$. Then diff above with $\dot{q}_\ell$ and we changing index j by i we have

$$\frac{\partial T}{\partial \dot{q}_\ell} = \sum_i a_{i\ell} \dot{q}_i + \sum_j a_{j\ell} \dot{q}_j$$

multiplying by $\dot{q}_\ell$ we have

$$\sum_\ell \dot{q}_\ell \frac{\partial T}{\partial \dot{q}_\ell} = \sum_{i,\ell} a_{i\ell} \dot{q}_i \dot{q}_\ell + \sum_{j,\ell} a_{i\ell} \dot{q}_j \dot{q}_\ell$$

Since all the indixes are dummy (artificial) and are symmetrical in i and j, thus the two Terms have the identical sum. Therefore

$$\sum_\ell \dot{q}_\ell \frac{\partial T}{\partial \dot{q}_\ell} = 2\sum_{i\ell} a_{i\ell} \dot{q}_i \dot{q}_\ell$$

But from eq ④ we have $T = \sum_{i\ell} a_{i\ell} \dot{q}_i \dot{q}_\ell$

Then putting its value we have

$$\sum_\ell \dot{q}_\ell \frac{\partial T}{\partial \dot{q}_\ell} = 2T.$$

In this case $V = 0$ for the Force. Then

$$L = T - V \Rightarrow L = T - 0 \Rightarrow L = T$$

Then

$$T = \frac{1}{2} \sum_\ell \dot{q}_\ell \frac{\partial T}{\partial \dot{q}_\ell} = \frac{1}{2} \sum_\ell \dot{q}_\ell \frac{\partial L}{\partial \dot{q}_\ell}$$

$$T = \frac{1}{2} \sum_\ell \dot{q}_\ell P_\ell \qquad \left(\text{As } \frac{\partial L}{\partial \dot{q}_\ell} = P_\ell\right)$$

$$\Rightarrow \boxed{T = \frac{1}{2} \sum_\ell \dot{q}_\ell P_\ell}$$

Proved the required Result.

SUMMETRIES AND CONSERVATION LAWS :-

we have Lagrangain which is a function of displacement, velocity and Time

∴ $L = L(q_j, \dot{q}_j, t)$

If L does not depends explicitly on Time Then

$$\frac{\partial L}{\partial t} = 0$$

The Total Time derivative of L is given by

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

As $\frac{\partial L}{\partial t} = 0$ because it is Time independent

Then $\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$

we have $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}\right)$

putting this value in above eq we have

$$\frac{dL}{dt} = \sum_j \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

or $\frac{dL}{dt} = \sum_j \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right]$

or $\frac{d}{dt}\left[\sum \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right] = 0$

The quantity in Brackets is Constant in Time

If we done $\sum_j \left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right)$ equal to H

As $H = \sum_j \left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right)$ — Ⓐ

where H is called Hamiltonian of The system

Since potential Energy does not depends on velocity or Time

$$V = V(q_j)$$

18-02-06

Then $\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T-V) = \frac{\partial T}{\partial \dot{q}_j} \cancelto{0}{\frac{\partial V}{\partial \dot{q}_j}}$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = P_j$$

(As $P_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}$)

Then The Equation Ⓐ becomes

$$H = \sum_j (P_j \dot{q}_j - L)$$

Then we have putt in above (∵ $\sum_l \dot{q}_l P_l = 2T$)

Then $H = \sum(2T - L) = (2T - L)$

or $H = 2T - (T - V)$ As $L = T - V$

$H = 2T - T + V$

or $\boxed{H = T + V = E}$

The Hamiltenian is equal to The Total energy of The system it is Constant of time

In equation Ⓐ we have

$\frac{d}{dt}\left(\sum P_j \dot{q}_j - L\right) = 0$ (As $H = \sum P_j \dot{q}_j - L$)

Then we have

$\frac{dH}{dt} = 0$ H is Constant

Hence The Hamiltonian is Constant.

* *

Written By

0345-9798233   Mati Ullah Surrani
Physics Deparatment
UOP.
ON 18Th Feb 2006

VELOCITY DEPENDENT POTENTIALS OF ELECTROMAGNETIC FIELDS:-

we can obtain the Lagrange's Equation in the Form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

even if the system is not Conservative in usual sense provided the generalize forces expressible in the form

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \text{ --- Ⓐ}$$

where $U(q_j, \dot{q}_j)$ is called generalized potential or velocity dependent potential.

"For particular set of Equation of motion, there is no unique choice of Lagrangian, such as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \text{ ---}$$

If the Lagrangain $L(q, \dot{q}, t)$ is an Appropertate Lagrangain and $E(q, t)$ is any differentable function, then

$$L'(q_j, \dot{q}_j, t) = L(q, \dot{q}, t) + \frac{dF}{dt}$$

If $\frac{dF}{dt} = 0$ then it does not depends on time when it depends on time then it will exist.

The Maxwell's Equation are

$$\bar{\nabla} \times \bar{E} = -\frac{1}{C}\frac{\partial B}{\partial t}, \quad \nabla . D = 4\pi\rho, \quad \nabla \times H = \frac{1}{C}\frac{\partial \bar{D}}{\partial t} + \frac{4\pi J}{c}$$

and $\nabla . B = 0$

In M.K.S system we have

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

or

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

As $\bar{B} = \nabla \times \bar{A}$ then where A is vector potential

18-02-06

$$\nabla \times E + \frac{\partial}{\partial t}(\nabla \times A) = 0$$

or

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

we can write this as

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

The Force on a charged particle having charge 'q' and moving in an electromagnetic field is given by

$$F = qE + q(\bar{V} \times B) \text{ called Lorents force}$$

$$F = q(E + \bar{V} \times \bar{B})$$

$$\text{or } F = q(E + \bar{V} \times \nabla \times A). \qquad \text{As } \bar{B} = \nabla \times A$$

putting the value of E we have

$$F = q\left(-\nabla\phi - \frac{\partial A}{\partial t} + (\bar{V} \times \bar{\nabla} \times A)\right)$$

$$\text{or } F = q\left(-\nabla\phi + (\bar{V} \times \bar{\nabla} \times A) - \frac{\partial A}{\partial t}\right) \text{ --- ①}$$

In order to find that which part of this Equation can be expressed as a gredient of the potential "U" we can evaluate this Equation Explicitly Considering only the x-component we have

$$(\nabla\phi)_x = \frac{\partial \phi}{\partial x} \text{ --- ②}$$

we do all the calculation and then take x-componts along x-axis

$$(\bar{V} \times (\bar{\nabla} \times A))_x = V_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - V_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

we add a term $V_x\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x}\right)$ we get

$$= V_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - V_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + V_x\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x}\right)$$

$$= V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_y}{\partial x} + V_z\frac{\partial A_z}{\partial x} - \left(V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_x}{\partial y} + V_z\frac{\partial A_x}{\partial z}\right) \text{ --- ②a}$$

Further the Total derivative of A_x is

$$A_x = A_x(x, y, z, t)$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} + \frac{\partial A_x}{\partial t}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}V_x + \frac{\partial A_x}{\partial y}V_y + \frac{\partial A_x}{\partial z}V_z + \frac{\partial A_x}{\partial t}$$

or $$\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} = \frac{\partial A_x}{\partial x}V_x + \frac{\partial A_x}{\partial y}V_y + \frac{\partial A_x}{\partial z}V_z$$

putting this value in eq ② we have

$$(\bar{V}\times(\bar{\nabla}\times\bar{A}))_x = -\left(\frac{d}{dt}A_x - \frac{\partial A_x}{\partial t}\right) + V_x\left(\frac{\partial A_x}{\partial x}\right) + V_y\left(\frac{\partial A_y}{\partial x}\right) + V_z\left(\frac{\partial A_z}{\partial x}\right)$$

$$(\bar{V}\times(\bar{\nabla}\times\bar{A}))_x = -\frac{d}{dt}A_x + \frac{\partial}{\partial t}A_x + \frac{\partial}{\partial x}(\bar{V}.\bar{A})$$

As $V.A = V_xA_x + V_yA_y + V_zA_z$

& $$\frac{\partial}{\partial x}(V.A) = \left(\frac{\partial}{\partial x}A_x\right)V_x + V_y\frac{\partial A_y}{\partial x} + V_z\frac{\partial A_z}{\partial x}$$

or $$(\bar{V}\times(\bar{\nabla}\times\bar{A}))_x = \frac{\partial}{\partial x}(\bar{V}.\bar{A}) + \frac{\partial}{\partial t}A_x - \frac{d}{dt}A_x$$

& $$(\nabla\phi)_x = \partial\phi/\partial x$$

putting these values in eq ① we have

$$F = q\left(-\nabla\phi + \bar{V}\times(\bar{\nabla}\times\bar{A}) - \frac{\partial A_x}{\partial t}\right)$$

$$= q\left(-\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial x}(\bar{V}.\bar{A}) + \cancel{\frac{\partial}{\partial t}A_x} - \frac{d}{dt}A_x - \cancel{\frac{\partial A_x}{\partial t}}\right)$$

or $$F = q\left(-\frac{\partial}{\partial x}(\phi - (\bar{V}.\bar{A})) - \frac{d}{dt}\frac{\partial}{\partial V_x}(\bar{V}.\bar{A})\right)$$

As $(V.A)_x = V_xA_x$

$$\frac{\partial}{\partial V_x}(V.A)_x = \frac{\partial V_x}{\partial V_x}A_x = A_x$$

Hence $$\frac{d}{dt}\frac{\partial}{\partial V_x}(V.A)_x = \frac{dA_x}{dt}$$

$$F = q\left(-\frac{\partial}{\partial x}(\phi - (\bar{V}.\bar{A})) - \frac{d}{dt}\frac{\partial}{\partial V_x}(\bar{V}.\bar{A})\right)$$

$$= q\left(-\frac{\partial}{\partial x}(\phi - \bar{V}.\bar{A})\right) + \frac{d}{dt}\frac{\partial}{\partial V_x}(\phi - \bar{V}.\bar{A})$$

18-02-06

As ϕ is not a function of V_x it is constant Hence its derivative will be zero

$$F_i = \frac{-\partial}{\partial x}\left(q.(\phi - \bar{V}.\bar{A})\right) + \frac{d}{dt}\frac{\partial}{\partial V_x}\left(q.(\phi - \bar{V}.\bar{A})\right) \quad ③$$

Comparing with the force eq Ⓐ

$$-\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} = Q_j = F$$

By comparing both the forces we can get the potential $U = q(\phi - \bar{V}.\bar{A})$

With this a charged particle move with uniform velocity 'V' in a electromagnetic field is given

$$L = T - U = T - q(\phi - \bar{V}.\bar{A})$$

where T is kinetic Energy of the Particle

* * *

The assignment and problems of classical mechanics are more (ness a) necessary Try to solve it 18 Feberuary Saturday 2006 ON 3:00 A.M DURING The Holidays of Universities due to

حضورؐ کی بے حرمتی میں شائع شدہ خاکے کی احتجاج کی وجہ سے 3 دن کی

16 فروری سے 19 فروری تک From vecations

* * *

DIFFICULTIES DUE TO CONSTRIANTS:-

Constraints introduce two types of difficulties in the solution of mechanical problem. (first the coordinate r_i is no longer all independent since they are connected by the equation of constraints hence the equations of motion are not all independent. The second is the forces of constraints the forces that will exert on the bead (or the gas particles) is not furnished they are among the unknown of the Problem and must be obtained from the solution we seek.

In case of holonomic constraints the 1st difficulty is solved by the introduction of generalized coordinates

A system of N Particle free from Constraints have 3N degree of freedom or 3N independent coordinate If there is exist holonomic constraints 'K' expressed in 'K' Equation then the degree of freedom in this case will be ~~3N~~-K where 'K' is the Number of ~~holonomic~~ exact constraints

* **HOME ASSIGNMENT**

Problem NO.13 A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is $F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right)$ where r is the distance of the particle to the center of force. Find the Generalized Potential that will result in such a force, and from the Lagrangian from the motion in a plane (The expression for F represent the force between two charges in weber' Electrodynamics.

SOLUTION :-

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right) \qquad U = ?$$

Consider

$$U = \frac{1}{r}\left(1 + \frac{v^2}{c^2}\right) = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right) \quad \because \; v = \dot{r}$$

$$U = \frac{1}{r} + \frac{\dot{r}^2}{rc^2}$$

This potential is called as weber potential in weber Electrodynamics

$$U = \frac{1}{r} + \frac{\dot{r}^2}{rc^2} \quad \text{— Ⓐ}$$

Then

$$\frac{\partial U}{\partial r} = -\frac{1}{r^2} - \frac{\dot{r}^2}{r^2c^2} \quad \text{— ⓐ}$$

and diff eq Ⓐ w.r.t to $\dot{r}$ we have

$$F = Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial t}$$

$$\frac{\partial U}{\partial \dot{r}} = \frac{2\dot{r}}{rc^2}$$

$$\text{or} \quad \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{d}{dt}\left(\frac{2\dot{r}}{rc^2}\right) = \frac{2\ddot{r}}{rc^2} + \frac{2\dot{r}}{c}\frac{d}{dt}\left(\frac{1}{r}\right)\frac{dr}{dt}$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{2\ddot{r}}{rc^2} - \frac{2\dot{r}^2}{r^2c^2} \quad \text{— ⓑ}$$

As we know that Lagrange's Equation form in this case is $\frac{d}{dt}\frac{\partial U}{\partial \dot{r}} - \frac{\partial U}{\partial r} = Q_j$

putting the values from eq ⓐ & eq ⓑ we have

$$\frac{2\ddot{r}}{rc^2} - \frac{2\dot{r}^2}{r^2c^2} - \left(-\left(\frac{1}{r^2} + \frac{\dot{r}^2}{r^2c^2}\right)\right) = Q_j$$

$$\frac{2\ddot{r}}{rc^2} - \frac{2\dot{r}^2}{r^2c^2} + \frac{1}{r^2} + \frac{\dot{r}^2}{r^2c^2} = Q_j$$

$$\frac{1}{r^2} - \frac{\dot{r}^2}{r^2c^2} + \frac{2\ddot{r}}{rc^2} = Q_j$$

$$\text{or} \quad \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right) = Q_j \rightarrow F$$

This is the Equation which is obtained, so the generalized Potential, resulting in the given force is

$$U = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right)$$

which is required result.

* *

MATI ULLAH SURANI. 0345 9798233

PROBLEM :-(HA) show that in thermodynamics internal energy $E(S,V)$ Free energy $P(V,T)$, GIBBS function $G(P,T)$ and enthalpy $H(S,P)$ are related with each other through the Legender Transformation

SOLUTION :-

Internal Energy $E = E(S,V)$, Entalphy $H = H(S,P)$

Free Energy $P = P(T,V)$ GIBBS function $G = G(T,P)$

The first two are related by haveing common variable "V" and the second two are related by having common variable P. Similarly like wise the 1st line two are related by S (entropy) and the 2nd line two are related by T (Temperature)

So all the four are related. Now we use the Legender Transformation.

Consider a function of only two variables $f(x,y)$ so that the differential of f has the form

$$df = u\,dx + v\,dy \quad ① \Rightarrow u = \frac{\partial f}{\partial x},\ v = \frac{\partial f}{\partial y}$$

we wish now to change the basis of description from x, y to a new distinct set of variable u and y so that differential quantities are expressed in term of dy and du.

Let g be a function of u and y defined by the equation

$$g = f - ux \quad ②$$

$$dg = df - d(ux) = df - u\,dx - x\,du$$

putting value of df from above we have

$$dg = u\,dx + v\,dy - u\,dx - x\,du = v\,dy - x\,du$$

which is desired from the quantity x and v are function of the variable u and y given by

$$x = -\frac{\partial g}{\partial u},\quad v = \frac{\partial g}{\partial y}$$

The Legender Transformation defined is frequently used in the thermodynamics.

12-04-06

The Enthalphy $H(S,P)$ is a function of entropy S and pressure P $\quad H = H(S,P)$

$$dH = \frac{\partial H}{\partial S}dS + \frac{\partial H}{\partial P}dP$$

where $T = \frac{\partial H}{\partial S}$ $\quad$ or $\quad dH = TdS + VdP \quad ①$

$V = \frac{\partial H}{\partial P}$

where T and V are Temperature and volume respectively. The Legender Transformation for GIBBS function is $\quad G = G(T,P)$

$$dG = \frac{\partial G}{\partial T}dT + \frac{\partial G}{\partial P}dP$$

where $S = \frac{\partial G}{\partial T}$

$V = \frac{\partial G}{\partial P}$ $\quad dG = SdT + VdP$

Generally we have $\quad G = H - TS$

on differentiating $\quad dG = d(H - TS) = dH - d(TS)$

$$dG = dH - TdS - SdT$$

putting value of d.H. from eq ① we get

$$dG = \cancel{TdS} + VdP - \cancel{TdS} - SdT$$

$$dG = VdP - SdT$$

Here $V = \frac{\partial G}{\partial P}$ and $S = -\frac{\partial G}{\partial T}$

To met with above equation

Similarly Internal Energy $\quad E = E(S,V)$

$$dE = \frac{\partial E}{\partial S}dS + \frac{\partial E}{\partial V}dV$$

$F = \frac{\partial E}{\partial S}$

$N = \frac{\partial E}{\partial V}$ $\quad \boxed{dE = FdS + NdV}$

Similarly for the Free energy

$$P = P(T,V)$$

where $R = \frac{\partial P}{\partial T}$ $\quad dP = \frac{\partial P}{\partial T}dT + \frac{\partial P}{\partial V}dV$

$$dP = RdT + AdV$$

$A = \frac{\partial P}{\partial V}$

The Transformation is actually done by the below equation

$$H_{(q,p,t)} = q_i p_i - L(q,\dot{q},t)$$

See page 341 Goldstein. P.T.O. # (74)

The Transformation from $(q_i, \dot{q}_i, t)$ to (q_i, p_i, t) differ from the type considered in eq ① to eq ② only in that more than one variable is to be transformed. In place of Lagrangian we deals with a function defined in eq ② except for minus sign

$$\underset{(q_i,p_i,t)}{H} = \dot{q}_i p_i - L(q_i, \dot{q}_i, t)$$

(where of course the summation convention has been implayed) Here H is known as the Hamiltonian

* ——— *

Written By 0345-9798233
Mati Ullah Surrani
Physics Deptt
University of Peshawar
Class No. 11 (Morning)
Surrani Bannu

فون: 5703445
عوامی فوٹو سٹیٹ
میٹرک، ایف اے، ایف ایس سی
بی اے، بی ایس سی، ایم۔ اے، ایم ایس سی، کے نوٹس دستیاب ہیں۔
محمد علی مارکیٹ کافی شاپ بازار پشاور یونیورسٹی
0321-9103972
0345-9223209
پروپرائٹر: جان صاحب

30.01.06

RAYLEIGH'S DISSIPATION FUNCTION:-

If we have a system in which all the forces acting are derivable from a scalar potential function then

conservative system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad \text{where } L = T - V \text{ and } V = V(q_j)$$

If all the forces are not derivable from a scalar function, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j \quad \text{—①} \quad \text{where } L = T - V,\ V = V(q_j)$$

$V = 0$ $L = T$

Where in eq ① we have L which have forces derivable from a scalar Potential V and Q_j represent those forces which are not derivable from a scalar Potential. An illustration of eq ① is the frictional force $\because F_{friction} \neq -\nabla\phi$ many times we come across a situation in which frictional forces is propertional to velocity $F_{friction} = -kV^n$ then

$$F_{fx} = -K_x V_x \quad \text{—②}$$

Where K_x is the x-component of the frictional force per unit velocity in the x-direction

$$\Rightarrow \quad F_{fx}/V_x = -K_x$$

Forces of such type are derivable from the Rayligh's Dissipation function $\mathcal{F}$, defined as

$$\mathcal{F} = \frac{1}{2}\sum_i \left[K_x V_{ix}^2 + K_y V_{iy}^2 + K_z V_{iz}^2\right] \quad \text{—③}$$

This represent the Energy.

where the frictional force is

$$F_f = -\bar{\nabla}_v \mathcal{F} \quad \text{(symbolically)}$$

It is only symbolic representation. we have introduced a new symbol

$$\nabla_v = \hat{i}\frac{\partial}{\partial V_x} + \hat{j}\frac{\partial}{\partial V_y} + \hat{k}\frac{\partial}{\partial V_z}$$

30.01.06

which is the vector velocity differential operator.

Physical Significance :-

To explain the physical significance of the Rayleigh's function, let us calculate the work done by the system against the frictional force is $dw_f = -\vec{F}_f \cdot d\vec{r}$

-ve sign represent that work is done by the system.

As $v = d\vec{r}/dt \Rightarrow d\vec{r} = v\,dt$ then

$$dw_f = -\vec{F}_f \cdot v\,dt$$

putting the value of $\vec{F}_f$ we have

$$dw_f = -(-(K_x V_x + K_y V_y + K_z V_z)) \cdot (V_x + V_y + V_z)\,dt$$

$$dw_f = (K_x V_x^2 + K_y V_y^2 + K_z V_z^2)\,dt$$

$$\Rightarrow \frac{dw_f}{dt} = (K_x V_x^2 + K_y V_y^2 + K_z V_z^2)$$

where $\frac{dw_f}{dt}$ is the rate of Energy dissipation function and which is equal to the twice of Rayleigh's function.

$$\frac{dw_f}{dt} = (K_x V_x^2 + K_y V_y^2 + K_z V_z^2) = 2F \sim ④$$

The Component of frictional force Q_j is given by

$$Q_j = \sum_i \vec{F}_{fi} \frac{\partial r_i}{\partial q_j}$$

putting $\vec{F}_{fi} = -\nabla_v F$ Then

$$Q_j = \sum -\nabla_v F \frac{\partial r_i}{\partial q_j}$$

and × the L.H.S by $\frac{d}{dt}$

$$= \sum -\nabla_v F \frac{\frac{d}{dt}\partial r_i}{\frac{d}{dt}\partial q_i}$$

$$= \sum -\nabla_v F \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

$$Q_j = -\sum \frac{\partial F}{\partial \dot{q}_j} (\nabla_v V_i)$$

In term of x
$\nabla_v V = \frac{\partial}{\partial V_x}(V_x) = 1$

$$Q_j = -\sum \frac{\partial F}{\partial \dot{q}_j} (1) = -\sum \frac{\partial F}{\partial \dot{q}_j} \quad — ⑤$$

From eq ① and eq ⑤ we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$$

when we are Considering the frictional force

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial F}{\partial \dot{q}_j}$$

or

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial F}{\partial \dot{q}_j} = 0 \quad — ⑥$$

Thus if the frictional forces are present, we must specify the two scalar function i-e the Lagrangian L and Rayleigh's dissipation function F to derive the equations of motion.

* * *

Application of Lagrange's Equation:-

1 Single particle in Space

(a) Motion of Particle in Cartesian Coordinate:-

Thus in this case we have three degree's of freedom along x-axis, along y-axis and along z-axis. As there is no interrept of the particle the body have only T the potential Energy V is zero.

i.e. K.E is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad — ①$$

When $V = 0$ Then $T = L - 0 \;=\; T = L$

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$$

and the derivative w.r.t x we will h

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right) = m\dot{x}$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} \;,\; \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

31.01.06

Then we have Lagrange' equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q_x \Rightarrow \frac{d}{dt} m\dot{x} = Q_x$$

$$\& \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = Q_y \Rightarrow \frac{d}{dt} m\dot{y} = Q_y$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = Q_z \Rightarrow \frac{d}{dt} m\dot{z} = Q_z$$

$$\Rightarrow m\ddot{x} = Q_x, \quad m\ddot{y} = Q_y, \quad m\ddot{z} = Q_z$$

Now since $r = x\hat{i} + y\hat{j} + z\hat{k}$

$$Q_x = F \cdot \frac{\partial r}{\partial x} = F \cdot \hat{i} = F_x$$

$$\Rightarrow F_x = Q_x = m\ddot{x}, \quad F_y = Q_y = m\ddot{y}, \quad F_z = Q_z = m\ddot{z}$$

Then

$$F = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$= m\ddot{x}\hat{i} + m\ddot{y}\hat{j} + m\ddot{z}\hat{k}$$

$$= m(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k})$$

$$= m(\ddot{\vec{r}})$$

$$F = ma$$

which comes back to the Newton 2nd law.

* *

(b) For plane polar coordinates:-

The Transformation Equation for plane polar coordinates is $x = r\cos\theta$, $y = r\sin\theta$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

we know that the K.E is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left((\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2 + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2\right)$$

$$= \frac{1}{2}m\Big(\dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2 - \cancel{2\dot{r}\cos\theta\, r\sin\theta\,\dot{\theta}} + \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{r}r\,\cancel{\dot{\theta}\sin\theta\cos\theta}\Big)$$

$$= \frac{1}{2}m\left(\dot{r}^2\cancel{(\cos^2\theta + \sin^2\theta)} + r^2\dot{\theta}^2\cancel{(\cos^2\theta + \sin^2\theta)}\right)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

31.01.06

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad —①$$

For Lagrangian we have

$$L = T - V$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V_{(r,\theta)} \quad —Ⓐ$$

Then

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m\frac{\partial}{\partial \dot{r}}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\partial}{\partial \dot{r}}(V_{(r,\theta)})$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\& \quad \frac{\partial L}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\right) - \frac{\partial}{\partial r}(V_{(r,\theta)})$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

Then we put these values in Lagrangian eq we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\text{or} \quad m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r} \qquad \left(\text{As } -\frac{\partial V}{\partial r} = F_r\right)$$

$$F_r = m\ddot{r} - mr\dot{\theta}^2$$

When $F_r = -\frac{k}{r^2}$ Then it is gravitational

If it is Columbic Then $F_r = \mp\frac{k}{r^2}$

$$F_r = -\frac{k}{r^2}$$

$$\frac{dV}{dr} = -\frac{k}{r^2} \Rightarrow dV = \frac{k}{r^2}dr$$

$$\Rightarrow V = \int \frac{k}{r^2}dr = -\frac{k}{r}$$

* *

For θ and $\dot{\theta}$ Differenting eqⒶ w.r.t θ and $\dot{\theta}$ respectively

$$\frac{\partial L}{\partial \theta} = \cancel{mr^2\dot{\theta}} - \frac{\partial V}{\partial \theta}$$

$$\& \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2\dot{r}r\dot{\theta}$$

31. 01. 06

Then putting The values we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = \frac{\partial V}{\partial \theta}$$

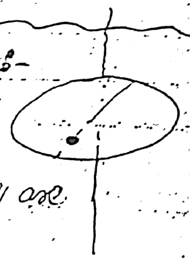
$$\Rightarrow m r^2\ddot{\theta} + 2\overset{m}{\dot{r}} r\dot{\theta} = \frac{\partial V}{\partial \theta}$$

$$\Rightarrow m r^2\ddot{\theta} + 2\overset{m}{\dot{r}} r\dot{\theta} = F_\theta .$$

Which gives The Angular Momentum.

* *

② Bead Sliding on a rotating wire:-



The motion of The bead is time dependent, so in this case time is used nessarily The Transformation eq are

$x = r\cos\theta = r\cos wt$

$y = r\sin\theta = r\sin wt$

Where ω is The uniform angular velocity through Which The bead is rotating.

$\dot{x} = \dot{r}\cos wt - r\sin wt(\omega)$

$\dot{y} = \dot{r}\sin wt + r\cos wt\ (\omega)$

Then we have The Expression for T

$$\therefore T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[(\dot{r}\cos wt + r\sin wt(\omega))^2 + (\dot{r}\sin wt + r\cos(wt)\omega)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2\cos^2 wt + r^2\omega^2\sin^2 wt + 2\dot{r}r\omega\,\cancel{\cos wt}\,\sin wt + \dot{r}^2\sin^2 wt + r^2\omega^2\cos^2 wt + 2\dot{r}r\omega\,\cancel{\cos wt}\,\sin wt\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2(\cos^2 wt + \sin^2 wt) + r^2\omega^2(\cos^2 wt + \sin^2 wt)\right]$$

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\omega^2\right]$$ The position of The bead in The wire is not fixed.

T is not quardatic homogenous function of $\dot{r}$

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\omega^2\right] \quad \text{—(A)}$$

31.01.06

Diff eq Ⓐ w.r.t. $\dot{r}$ we get As $\boxed{V=0}$ $(L=T)$

$$\frac{\partial L}{\partial \dot{r}} = \frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}m(\dot{r}^2 + r^2\omega^2)\right)$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \quad + \quad \frac{\partial T}{\partial r} = mr\omega^2$$

Then $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{d}{dt}m\dot{r} - mr\omega^2 = 0$$

$$m\ddot{r} - mr\omega^2 = 0$$

$$\Rightarrow m\ddot{r} = mr\omega^2$$

Case (I) when $\omega = \frac{V}{r}$ Then (As $F = m\ddot{r}$)

$$F = m\ddot{r} = mr\left(\frac{V}{r}\right)^2$$

$$= mr\frac{V^2}{r^2} = \frac{mV^2}{r}$$

$\Rightarrow F = \frac{mV^2}{r}$ which gives centrifugal force

Case (II) $\cancel{m}\ddot{r} = \cancel{m}r\omega^2$

$\Rightarrow \ddot{r} = r\omega^2$ which is called centrifugal accelartion. $a = r\omega^2$ $r(V/r)^2$

HOME ASSIGMENT

*

EXERCISE

This answer is not solved by sir. The Answer of This question is written before Velocity dependent potential $r, \dot{r}, \frac{V}{r}$

Q NO. 13 Gold stein # 33

A particle moves in a plane under The influence of a force, acting towards center of force, whose megnitude is

$$F = \frac{1}{r^2}\left[1 - \frac{(\dot{r}^2 - 2\ddot{r}r)}{c^2}\right]$$

where r is The ~~(radius)~~ distance of The Particle to The Center of force. Find The Generalized Potential that will result in such a force, and from That the Lagrangain for the motion in a plane (The expression for F represent the force between two charges in weber's electrodynamics.

Solution:- we are going out to find out The potential V which is required.

01.02.06

we Know That $F = -\frac{\partial V}{\partial r}$

or $F = -\frac{dV}{dr}$

$\Rightarrow -dV = +F dr$

$\Rightarrow dV = -F dr$

putting the value of F given

$$dV = -\frac{1}{r^2}\left[1 - \frac{(\dot{r}^2 - 2\ddot{r}r)}{c^2}\right] dr$$

$$dV = \left[-\frac{1}{r^2} + \frac{(\dot{r}^2 - 2\ddot{r}r)}{r^2c^2}\right] dr$$

$$dV = \frac{-1 dr}{r^2} + \frac{(\dot{r}^2)}{r^2c^2} dr - \frac{2\ddot{r}r}{r^2c^2} dr$$

Integrating both sides we get

$$\int dV = -\int \frac{1}{r^2} dr + \frac{\dot{r}^2}{c^2}\int \frac{1}{r^2} dr - \frac{2\ddot{r}}{c^2}\int \frac{1}{r} dr$$

$$V = -\left(-\frac{1}{r}\right) + \frac{\dot{r}^2}{c^2}\left(-\frac{1}{r}\right) - \frac{2\ddot{r}}{c^2}\ln r$$

$$V = \frac{1}{r} - \frac{\dot{r}^2}{rc^2} - \frac{2\ddot{r}}{c^2}\ln r$$

or $$V = \frac{1}{r}\left[1 - \frac{\dot{r}^2}{c^2} - \frac{2\ddot{r}r}{c^2}\ln r\right]$$

or $$V = \frac{1}{r}\left[1 - \frac{\dot{r}^2 - 2\ddot{r}r\ln r}{c^2}\right]$$

or $$V = \frac{1}{r}\left[1 - \frac{(\dot{r}^2 + 2\ddot{r}r\ln r)}{c^2}\right]$$

Which is the required geralized Potential

*

*

73

01.02.06

Equation of Constraints or degree of freedom = 3P-K where P is number of Particles and K is the number of Constraints on That Particle.

① For free Particle in Space :-

No. of Particle = P = 1

No. of Constraints on it is = 0

Then Equation of Constraints will be

$3(1) - 0 = 3$

⑪ For double Pendulum :-

Number of Particle = P = 2

Constraints on it is = K = 4

Equation of Constraints = 3P - K = 3(2) - 4

$3P - K = 6 - 4 = 2$

2006 * see notes *

PROBLEM NO. ~~4~~ ~~Goldstein~~ # 31 HOME ASSIGNMENT

Show That for a single Particle with Constant mass The Equation of motion implies the following differential Equation for The Kinetic Energy

$$\frac{dT}{dt} = F.v$$

$\frac{1}{2}mv^2$

while if The mass varies with time The Corresponding Equation is $\frac{d}{dt}(mT) = F.P$

SOLUTION :-

When There is a single Particle and its mass is Constant Then given Equation is

$$\frac{d}{dt}T = \bar{F}.\bar{V}$$

we Know That $T = \frac{1}{2}m\dot{x}^2$

Taking L.H.S $= \frac{d}{dt}T = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right)$

$= \frac{2}{2}m\dot{x}\frac{d}{dt}(\dot{x})$

(As $\ddot{x} = a$)

$= m\dot{x}\ddot{x}$

$= m\dot{x}a = ma\dot{x}$

$$\frac{d}{dt}T = m a \dot{x} \qquad \text{As } F = ma \quad (\dot{x} = v).$$

$$\frac{dT}{dt} = FV$$

$$\Rightarrow \frac{d}{dt}T = F.V.$$

which is The required solution when The mass is constant.

If The mass varies with time Then

$$\frac{d}{dt}(mT) \qquad \text{where } T = \frac{1}{2}m\dot{x}^2$$

$$\frac{d}{dt}\left(m\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\left(\frac{1}{2}m^2\dot{x}^2\right)$$

$$\frac{d}{dt}(mT) = \frac{2}{2}m^2\dot{x}^{2-1}\frac{d}{dt}(\dot{x})$$

$$= m^2\dot{x}\,\ddot{x}$$

$$= m\dot{x}\,ma$$

$$\frac{d}{dt}(mT) = ma.m\dot{x} \qquad (m\dot{x} = mv = \bar{P})$$

$$\Rightarrow \frac{d}{dt}(mT) = \bar{F}.\bar{P} \qquad (ma = m\ddot{x} = \bar{F})$$

This The other ~~Portion is~~ also solved.

* *

PROBLEM NO. 18 ~~#33~~ Goldstein

A particle of mass m moves in one dimension such That it has The Lagrangain

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V_{(x)} - V^2_{(x)}$$

Where V is some differentiable function of x. Find The Equation of motion for x(t) and describe The physical nature of The system on The basis of This Equation.

SOLUTION:- we have The Equation of motion of Lagrangian is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Here is one dimensional case, So That q_j will be replaced by x_j + $\dot{q}_j$ by $\dot{x}_j$ so, That

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 \quad \text{—①}$$

The Lagrangian is given such That

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V_{(x)} \mp V^2_{(x)} \quad \text{—Ⓐ}$$

Diff eq Ⓐ w.r.t x we have

$$\frac{\partial L}{\partial x} = m\dot{x}^2\frac{d}{dx}V_{(x)} + 2V_{(x)}\frac{d}{dx}V_{(x)}$$

Similarly Diff eq Ⓐ w.r.t $\dot{x}$ we have

$$\frac{\partial L}{\partial \dot{x}} = \frac{4m^2\dot{x}^3}{12} + 2m\dot{x}V_{(x)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\cancel{12}m^2\dot{x}^2\ddot{x}}{\cancel{12}} + 2m\ddot{x}V_{(x)} + 2\dot{x}\,m\frac{d}{dt}V_{(x)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} + 2\dot{x}m\frac{d}{dx}V_{(x)}.\frac{dx}{dt} \checkmark$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} + 2\dot{x}.\dot{x}m\frac{d}{dx}V_{(x)} \quad \text{Note this point}$$

$$= m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} + 2\dot{x}^2m\frac{d}{dx}V_{(x)}$$

putting The values of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$ + $\frac{\partial L}{\partial x}$ in eq ①

we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} + 2\dot{x}^2\frac{d}{dx}V_{(x)}$$

$$- m\dot{x}^2\frac{d}{dx}V_{(x)} + 2V_{(x)}\frac{d}{dx}V_{(x)}$$

$$= m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} + 2\dot{x}^2m\frac{dV_{(x)}}{dx} - m\dot{x}^2\frac{d}{dx}V_{(x)} + 2V_{(x)}\frac{d}{dx}V_{(x)}$$

$$= m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} - m\dot{x}^2\frac{dV_{(x)}}{dx} + 2V_{(x)}\frac{d}{dx}V_{(x)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_{(x)} + m\dot{x}^2\frac{d}{dx}V_{(x)} + 2V_{(x)}\frac{d}{dx}V_{(x)} = 0$$

$$m\ddot{x}(m\dot{x}^2 + 2V_{(x)}) + \frac{dV_{(x)}}{dx}(m\dot{x}^2 + 2V_{(x)}) = 0$$

$$\Rightarrow \quad (m\ddot{x} + \frac{d}{dx}V_{(x)})(m\dot{x}^2 + 2V_{(x)}) = 0$$

As we can write $F = -\frac{dV_{(x)}}{dx}$ and

$$T = \frac{1}{2}m\dot{x}^2 \Rightarrow 2T = m\dot{x}^2$$

Then above eq will be

$$(m\ddot{x} - (-\frac{d}{dx}V_{x}))(2T + 2V_{(x)}) = 0$$

$$(m\ddot{x} - F)2(T + V_{(x)}) = 0$$

Dividing by 2 we have

$$(m\ddot{x} - F)(T + V) = 0$$

$$(ma - F)(T + V) = 0$$

we can say That all times either The difference between F and "ma" is zero or The sum of K.E and P.E is zero.

This is The physical Intrepration also.

$\circledcirc$

PROBLEMS:- 10 Numbers. $\boxed{\text{HOME ASSIGNMENT}}$

A bead of mass "m" is Constrained to move along a smooth rigid wire having The shape of a hyperbola $xy = c$ so That The K.E may be Expressed as

$$T = \frac{1}{2}m\dot{x}^2(1 + \frac{c^2}{x^4})$$

Write down The Equation of motion and show That if y-axis is vertical, The generalized force Corresponding to x is given as

$\boxed{\text{Solution}}$ $\boxed{F_x = mg\,c/x^2}$

14.02.06

$\boxed{\text{SOLUTION}}$ The K.E of The bead is given as

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

As $z = 0$. Then we have

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad \text{—} ①$$

Now equation of motion of bead on surface is

$$xy = c \Rightarrow x = c/y \Rightarrow \dot{x} = \frac{-c}{y^2}\dot{y}$$

$$\Rightarrow \dot{x} = -\frac{c}{y^2}\dot{y}$$

$$y^2 = \frac{c^2}{x^2}$$

$$\dot{x} = \frac{-c}{y^2}\dot{y} \quad \text{Then}$$

$$\dot{x} = \frac{-c}{\frac{c^2}{x^2}}\dot{y} = \frac{-x^2}{c}\dot{y} \Rightarrow \dot{x} = -\frac{x^2}{c}\dot{y} \quad \text{—} \text{(1A)}$$

and $\dot{y} = -\frac{\dot{x}c}{x^2}$

$$\Rightarrow \dot{y}^2 = \frac{\dot{x}^2c^2}{x^4} \quad \text{—} ②$$

putting the values of (1A) and ② in eq① we get

$$T = \frac{1}{2}m(\dot{x}^2 + \frac{\dot{x}^2c^2}{x^4}) = \frac{1}{2}m(\dot{x}^2 + \frac{\dot{x}^2c^2}{x^4})$$

$$T = \frac{1}{2}m\dot{x}^2(1 + \frac{c^2}{x^4})$$

The P.E is given By $U = mgy$.

The Lagrangain will be

$$L = T - U$$

$$L = \frac{1}{2}m\dot{x}^2(1 + \frac{c^2}{x^4}) - mgy$$

then

$$\frac{\partial L}{\partial \dot{x}} = \frac{2}{2}m\dot{x}(1 + \frac{c^2}{x^4}) = m\dot{x}(1 + \frac{c^2}{x^4})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}(1 + \frac{c^2}{x^4}) - \frac{4m\dot{x}c^2}{x^5}$$

False

similarly

$$\frac{\partial L}{\partial x} = \frac{1}{2}m\dot{x}^2(-\frac{4c^2}{x^5}) = -\frac{2m\dot{x}^2c^2}{x^5}$$

20-02-06

As we have to find The Lagranges Equation of motion Then $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

putting all the values in it we get

$$m\ddot{x}\left(1+\frac{c^2}{x^4}\right)^x - \left(-\frac{2m\dot{x}^2c^2}{x^5}\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x}\left(1+\frac{c^2}{x^4}\right) + \frac{2m\dot{x}^2c^2}{x^5}$$

$$\Rightarrow m\ddot{x} + \frac{m\ddot{x}c^2}{x^4} + 2\frac{m\dot{x}^2c^2}{x^5} = 0$$

or $$m\ddot{x}x^5 + m\ddot{x}xc^2 + 2m\dot{x}^2c^2 = 0$$

$$\Rightarrow \boxed{m\ddot{x}x^5 + m\ddot{x}xc^2 + 2m\dot{x}^2c^2 = 0.}$$

This is The Equation of motion when The bead is moving on hyperbola.

Now The Generalize force will be given as

$F = -\nabla V$ Here $V = mgy$

$F_x = -\nabla_x\left(mg\frac{c}{x}\right)$ But $xy = c$

$y = \frac{c}{x}$

Then $V = mg\frac{c}{x}$

$$F_x = -\frac{\partial}{\partial x}\left(mg\frac{c}{x}\right)$$

$$= -mgc\frac{\partial}{\partial x}x^{-1}$$

$$= -(-1)\frac{mgc}{x^2} = \frac{mgc}{x^2}$$

$$\Rightarrow \boxed{F_x = \frac{mgc}{x^2}}$$

which is The required Generalized force corresponding to x

* *

HOME ASSIGNMENT 21-02-06

PROBLEM NO. 11 Show That Lagrange's Equation of motion $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$ Can also be written in The form

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

which is some time called The Nielson form of The Lagrange's Equation

SOLUTION:-

we know That

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

or $$\frac{\partial}{\partial \dot{q}_j}\frac{d}{dt}T - \frac{\partial T}{\partial q_j} = Q_j \qquad \left(\text{As } \frac{\partial T}{\partial t} = \dot{T}\right)$$

or $$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \text{ — Ⓐ}$$

putting The values in above equation we have

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}(2T) = Q_j$$

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial}{\partial q_j}T = Q_j$$

which is The required Neilson from of The Lagranges Equation

As $$T = \frac{1}{2}m_i\dot{r}^2$$

$$= \frac{1}{2}m_i\dot{r}\dot{r}$$

$$\frac{\partial T}{\partial q_j} = \frac{1}{2}m_i\frac{\partial}{\partial q_j}(\dot{r}\dot{r})$$

$$= \frac{1}{2}m_i\dot{r}\frac{\partial \dot{r}}{\partial q_j} + \frac{1}{2}m_i\dot{r}\frac{\partial \dot{r}}{\partial q_j}$$

$$\frac{\partial T}{\partial q_j} = m_i\dot{r}\frac{\partial \dot{r}}{\partial q_j}$$

$$= \frac{\partial}{\partial q_j}(m_i\dot{r}.\dot{r})$$

$$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}(m_i\dot{r}^2)$$

As $T = \frac{1}{2}m_i\dot{r}^2$

$\Rightarrow 2T = m_i\dot{r}^2$

puttin in abov

$$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}(2T)$$

14-02-06

Goldstein

CHAPTER NO.2 VARIATIONAL PRINCIPLES AND LAGRANGE'S EQUATIONS

CHAPTER-NO.11 TAKWALE # 287

HAMILTON'S Principle :-

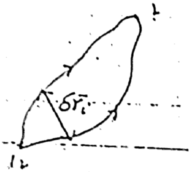
STATEMENT :- The motion of a system from t_1 to t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L\,dt \quad \text{where } L = T - V \text{ is an}$$

extremum for all paths motion

i-e

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0$$



PROOF :-

Consider a system of n-particle

The equation of motion of ith particle is

$$m_i \frac{d^2\bar{r}_i}{dt^2} = \bar{F}_i$$

Let the path of ith particle is varied by virtual displacement $\delta\bar{r}_i$ without changing the end points, then

$$m_i \frac{d^2\bar{r}_i}{dt^2} \cdot \delta\bar{r}_i = \bar{F}_i \cdot \delta\bar{r}_i$$

For the entire system

$$\sum_i m_i \frac{d^2\bar{r}_i}{dt^2} \cdot \delta\bar{r}_i = \sum_i \bar{F}_i \cdot \delta\bar{r}_i = \delta\omega \text{ ——} ①$$

Now Taking the K.E we have

$$T = \frac{1}{2} \sum_i m_i \frac{d\bar{r}_i}{dt} \cdot \frac{d\bar{r}_i}{dt}$$

The corresponding variation in K.E due to δr_i

$$\delta T = \frac{1}{2} \delta \sum_i \left[m_i \frac{d\bar{r}_i}{dt} \cdot \frac{d\bar{r}_i}{dt} \right]$$

20-02-06

$$\delta T = \sum_i \frac{1}{2} m_i \frac{d\bar{r}_i}{dt} \,\delta \frac{d\bar{r}_i}{dt} + \sum_i \frac{1}{2} m_i \frac{d\bar{r}_i}{dt} \,\delta \frac{d\bar{r}_i}{dt}$$

$$\delta T = \sum_i m_i \frac{d\bar{r}_i}{dt} \,\delta\left(\frac{d\bar{r}_i}{dt}\right) \text{ ——}$$

Now we consider $\delta T = \sum_i m_i \frac{d\bar{r}_i}{dt} \cdot \frac{d}{dt} \delta\bar{r}_i$ —— ②

$$\sum_i m_i \frac{d}{dt}\left[\frac{d\bar{r}_i}{dt} \cdot \delta\bar{r}_i\right] = \sum_i m_i \frac{d^2\bar{r}_i}{dt^2} \cdot \delta\bar{r}_i + m_i \frac{d\bar{r}_i}{dt} \frac{d}{dt}(\delta\bar{r}_i)$$

putting the values from eq ① & eq ②

$$\sum_i m_i \frac{d}{dt}\left[\frac{d\bar{r}_i}{dt} \cdot \delta\bar{r}_i\right] = \delta\omega + \delta T \text{ —— } ③$$

Integrating both sides from t_1 to t_2 we have

$$\sum_i \int_{t_1}^{t_2} m_i \frac{d}{dt}\left[\frac{d\bar{r}_i}{dt} \cdot \delta\bar{r}_i\right] dt = \int_{t_1}^{t_2} (\delta\omega + \delta T)\,dt$$

$$\sum_i \int_{t_1}^{t_2} d\left[m_i \left[\frac{d\bar{r}_i}{dt} \cdot \delta\bar{r}_i\right]\right] = \int_{t_1}^{t_2} (\delta\omega + \delta T)\,dt$$

$$m_i \frac{d\bar{r}_i}{dt} \cdot \delta\bar{r}_i \Big|_{t_1}^{t_2} = \delta \int_{t_1}^{t_2} (\omega + T)\,dt$$

As $\delta r_i = 0$ because the end points are fixed (not changed) there fore the L.H.S of the Equation will become zero

$$0 = \delta \int_{t_1}^{t_2} (\omega + T)\,dt$$

$$\text{or} \quad \delta \int_{t_1}^{t_2} (\omega + T)\,dt = 0$$

If force is conservative then

$$\omega = -V$$

Then putting in above

As $F = -\bar{\nabla} V$

$$\int \bar{F} \cdot d\bar{r} = -\int \nabla V \cdot dr$$

$$= -\int \nabla V \cdot dr$$

$$\Rightarrow \omega = -\int dV$$

$$\omega = -V$$

20-02-06

$$\delta\int_{t_1}^{t_2}(\omega+T)\,dt=0$$

putting $\omega=-V$ we get

$$\delta\int_{t_1}^{t_2}(-V+T)\,dt=0=\delta\int_{t_1}^{t_2}(T-V)\,dt$$

or $$\delta\int_{t_1}^{t_2}L\,dt=0$$

where $L=T-V$

This shows that $\int_{t_1}^{t_2}L\,dt$ has extremum values.

* * *

DERIVATION OF LAGRANGE'S EQUATION ON FROM HAMILTON'S PRINCIPLE

we know that the Hamilton principle is

$$\delta\int_{t_1}^{t_2}L\,dt=0 \quad \text{—①}$$

we have $L=L(q_j,\dot{q}_j,t)$

then the

$$\delta L=\sum_i\frac{\partial L}{\partial q_j}\delta q_i+\sum_i\frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_i+\frac{\partial L}{\partial t}dt \quad \text{—Ⓐ}$$

For conservative system Lagrangain does not depends on time $\frac{\partial L}{\partial t}=0$ Then eq Ⓐ ⟹

$$\int_{t_1}^{t_2}\delta L\,dt=\int_{t_1}^{t_2}\left(\sum_i\frac{\partial L}{\partial q_j}\delta q_i+\sum_i\frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_i\right)dt \quad \text{—②}$$

In eq ② consider the 2nd term on R.H.S

$$\sum_i\int_{t_1}^{t_2}\overset{\text{I}}{\left(\frac{\partial L}{\partial \dot{q}_j}\right)}\overset{\text{II}}{(\delta\dot{q}_i)}\,dt=\sum_i\left(\frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2}\delta\dot{q}_j\,dt-\int_{t_1}^{t_2}\left[\frac{d}{dt}\cdot\frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2}\delta\dot{q}_j\,dt\right]dt\right)$$

Integrating by parts

we have applied integration by parts.

21-09-06

$$\left(\text{As } \int uv\,dx=v\int u\,dx-\int\left(\frac{dv}{dt}\int u\right)dx\right)$$

$$=\sum_i\left[\frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2}\frac{d}{dt}(\delta q_j)\,dt-\int_{t_1}^{t_2}\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2}\frac{d}{dt}\delta q_j\,dt\right]dt\right.$$

$$=\sum_i\left[\frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2}d(\delta q_j)-\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\int_{t_1}^{t_2}d(\delta q_j)\right]dt\right.$$

$$=\frac{\partial L}{\partial \dot{q}_j}\Big[\delta q_j\Big|_{t_1}^{t_2}-\sum_i\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_i\,dt\right.$$

∵ t_1 and t_2 are fixed. Therefore δq_i is zero for the first integral

$$\sum_i\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_i\,dt=0-\sum_i\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_i\,dt \quad \text{—③}$$

From eq ② & eq ③ we have

$$\int_{t_1}^{t_2}\delta L\,dt=\sum_i\int_{t_1}^{t_2}\frac{\partial L}{\partial q_j}\delta q_i\,dt-\sum_i\int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\delta q_i\,dt$$

$$\int_{t_1}^{t_2}\delta L\,dt=\sum_i\left\{\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_j}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_i\,dt\right.$$

From eq ① we have

$\delta\int_{t_1}^{t_2}L\,dt=0$ Then above equation will be

$$\sum_i\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_j}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_i\,dt=0$$

In this case we have two chances.

Either $\int_{t_1}^{t_2}\delta q_i\,dt=0$ or $\neq 0$ Then

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}+\frac{\partial L}{\partial q_j}=0 \overset{\text{mul by }-1}{\Rightarrow} \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0}$$

21-02-06

which is The Derivation of Lagrangies Equation from Hamilton principle.

* *

Hamilton's Principle is Equivalent to Newton's Second law:–

OR Newton 2nd law of motion from Hamilton's Principle.

Let us suppose a particle of mass "m" at a position $r = r(x, y, z)$ is moving under the action force F, The kinetic energy of The particle is

$$T = \frac{1}{2} m \dot{r}^2$$

The variation in the work done is given by

$$\delta W = F.dr = -\delta V$$

$$\bar{F}.d\bar{r} = \delta W = -\delta V \quad — Ⓐ$$

where V is some Scalar Potential

$$\left\{ \begin{aligned} dw &= F.dr \\ dw &= -\nabla V.dr \\ dw &= -dV \\ dw &= -\delta V \end{aligned} \right.$$

From Hamilton's variational Principle we have

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} (T - V)\, dt = \int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0 \quad — ①$$

we know That K.E is

$$T = \frac{1}{2} m \dot{r}^2 = \frac{1}{2} m \dot{r}.\dot{r}$$

$$\delta T = \delta\left(\frac{1}{2} m \dot{r}\dot{r}\right) = \frac{1}{2} m\, \delta(\dot{r}.\dot{r})$$

$$= \frac{1}{2} m (\dot{r}.\delta\dot{r} + \dot{r}.\delta\dot{r}) = \frac{1}{2} m (2\dot{r}.\delta\dot{r})$$

$$\delta T = m\dot{r}.\delta\dot{r} \quad — Ⓑ$$

As we know That

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0$$

putting The values from Ⓐ & Ⓑ we have

85

22-02-06

$$\int_{t_1}^{t_2} (\delta T - \delta V) dt = \int_{t_1}^{t_2} (m\dot{r}.\delta\dot{r} + \vec{F}.\delta\bar{r})\, dt = 0 \quad — ②$$

In eq ② integrating 1st term by integration by Parts

$$= \int_{t_1}^{t_2} \overset{\text{I}}{m\dot{r}}\, \overset{\text{II}}{\delta\dot{r}}\, dt + \int_{t_1}^{t_2} \bar{F}.\delta\bar{r}\, dt = 0$$

$$= m\dot{r} \int_{t_1}^{t_2} \delta\dot{r}\, dt - \int_{t_1}^{t_2} \left[\frac{d}{dt}(m\dot{r}) \int_{t_1}^{t_2} \delta\dot{r}\, dt\right] dt + \int_{t_1}^{t_2} \bar{F}.\delta\bar{r}\, dt = 0$$

$$= m\dot{r} \int_{t_1}^{t_2} \frac{d}{dt} \delta r.dt - \int_{t_1}^{t_2} \left(m\ddot{r} \int_{t_1}^{t_2} \frac{d}{dt} \delta r\, dt\right) dt + \int_{t_1}^{t_2} \bar{F}.\delta\bar{r}\, dt = 0$$

$$= m\dot{r} \int_{t_1}^{t_2} d\delta\bar{r} - \int_{t_1}^{t_2} m\ddot{r}\delta r\, dt + \int_{t_1}^{t_2} \bar{F}.\delta\bar{r}\, dt = 0$$

$$= m\dot{r}\delta r \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{r}\delta r\, dt + \int_{t_1}^{t_2} \bar{F}.\delta\bar{r}\, dt = 0$$

∵ The t_1 and t_2 are fixed, Therefore $\delta r \Big|_{t_1}^{t_2} = 0$

Then we have

$$= -\int_{t_1}^{t_2} m\ddot{r}\,\delta r \cancel{dt} \int_{t_1}^{t_2} F.\delta r\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} (F - m\ddot{r})\, \delta\bar{r}\, dt = 0$$

Since δr vanishes at end points, The above equation is True for every virtual displacement (variational) Therefore

$$F - m\ddot{r} = 0$$

$$\Rightarrow F = m\ddot{r} = ma$$

which is second law of Newton.

* *

22-02-06

when Lagrange's Equation is derived from the Hamilton principle then it is some time called as Euler-Lagrange's Equation.

*

SOME TECHNIQUES OF THE CULCULUS OF VARIATIONS :-

The basic Problem of calculus of variation is to find a path $y = y(x)$ (in one dimension) between the Two Points x_1 and x_2, such That the time integral of Some function

$$f = f(y, \dot{y}, x) \qquad \text{where } y = y(x)$$

where $\dot{y} = dy/dx$ is having a stationary value (extremum) i-e $\quad \dot{y} = \frac{dy}{dx}$

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx \text{ --- } ①$$

This integral has stationary value relative to The path, differing infinitesimaly from the correct function $y(x)$

Let (x_1, y_1) and (x_2, y_2) be Two points in The space we shown figure 1, Two varied Paths between Two points

$y_1 = y(x_1)$ and $y_2 = y(x_2)$

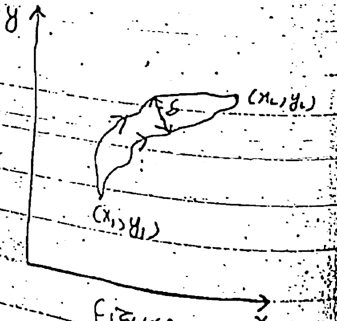


figure 1

In order to find a path or paths which would enables us to give an extremum value for The integral

We state The problem in The languge of differential culculus, for This we associate a parameter "α" with all The possible curves or Paths.

27.02.06

The Parameter α should be such That for Some value of α, say $\alpha = 0$, The curve under consideration (examination) would concide with The path That would gives The extremum value (correct path)

Then y will be a function of both The independent variable x and The parameter α we can write as

$$y(x, \alpha) = y(0, x) + \alpha\, \eta(x) \text{ --- } ②$$

where $\eta(x)$ is some function of x which has Continous first order and second order derivatives (continous and well behaved) and function vanishes at both $x = x_1$ and $x = x_2$

The condition $\eta(x_1) = \eta(x_2) = 0$ ensures that The varied function $y(\alpha, x)$ will be identical to $y(x)$ at The extremities of The path.

With The dependence of 'y' on 'α' and in addition to x, The integral become a function of The parameter α

eq ① ⇒

$$J(\alpha) = \int_{x_1}^{x_2} f\,[y(\alpha, x), \dot{y}(\alpha, x); x]\, dx \text{ -- } ③$$

Then The condition That $J(\alpha)$ has an extremum value is

$$\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0 \quad \text{or} \quad \left[\frac{dJ}{d\alpha}\right] = 0 \text{ --- } ④$$

Differentiating eq ③ under integral sign,

$$\frac{dJ}{d\alpha} = \frac{d}{d\alpha} \int_{x_1}^{x_2} f\,[y(\alpha, x), \dot{y}(\alpha, x), x]\, dx$$

27.02.06

$$\text{or } \frac{dJ}{d\alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right]dx \qquad \text{As } \dot{y} = \frac{\partial y}{\partial x} = \frac{dy}{dx}$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\cdot\frac{\partial(\partial y/\partial x)}{\partial \alpha}\right]dx$$

$$\text{or } \frac{dJ}{d\alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha}\right]dx \quad \text{—⑤}$$

Integrate the second term of eq ⑤ by parts we have

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha}\,dx = \int_{x_1}^{x_2}\overset{I}{\frac{\partial f}{\partial \dot{y}}}\;\overset{II}{\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \alpha}\right)}\,dx$$

$$= \frac{\partial f}{\partial \dot{y}}\int_{x_1}^{x_1}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \alpha}\right)dx - \int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \alpha}\right)dx\right]dx$$

$$= \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\left(\frac{\partial y}{\partial \alpha}\right)dx \quad \text{—⑥}$$

But since we have $\frac{\partial y}{\partial \alpha} = \eta(x)$ from eq ②

then

$$\frac{\partial y}{\partial \alpha}\Big|_{x_1}^{x_2} = \eta(x)\Big|_{x_1}^{x_2} = \eta(x_2) - \eta(x_1) = 0$$

No variation at end points so first term in eq ⑥ is zero

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha} = -\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\left(\frac{\partial y}{\partial \alpha}\right)dx \quad \text{—⑦}$$

putting the value of eq ⑦ in eq ⑤ we have

$$\left[\frac{dJ}{d\alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\left(\frac{\partial y}{\partial \alpha}\right)dx\right]dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right]\left(\frac{\partial y}{\partial \alpha}\right)dx$$

27-02-06

Then we have

$$\left[\frac{dJ}{d\alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\eta(x)\,dx \quad \text{—⑧}$$

$$\text{where } \eta(x) = \left(\frac{\partial y}{\partial \alpha}\right)$$

Equation ⑧ appears to be independent of "α", but is not like this, since y and ẏ are function of α

when α is zero then

$$y(\alpha, x) = y(x)$$

and the dependence (of α) on α disappear since η(x) is an arbitrary function because $\eta(x_1) = \eta(x_2) = 0$

the integrand must vanish for α = 0, Thus we have

$$\left[\frac{dJ}{d\alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\eta(x)\,dx = 0$$

From this we have

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0} \quad \text{—⑨}$$

y, ẏ are original functions α = 0

Equation ⑨ is called Euler's Equation and represent a necessary condition that the integral J has the extremum value.

* *

MATI ULLAH 0345-9798233
M.Sc (Prevous)
Physics Deparatment U.O.P
Village Jahangir Lalozai P/o Torka
Surrani Bannu (N.W.F.P)

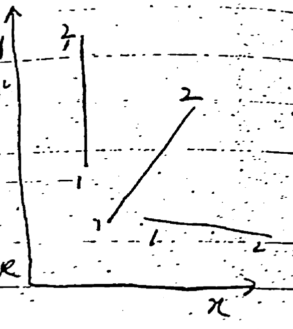
01-03-06

SIMPLE APPLICATION OF CALCULUS OF VARIATION:

① Shortest distance between Two Points IN A PLANE:

The element of length in the xy plane is given as

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$



The total length I in the plane is

$$I = \int^{2} ds$$

putting the value of ds we have

$$I = \int_{x_1}^{x_2} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\left(\text{As } \dot{y} = \frac{dy}{dx}\right)$$

Then

$$I = \int_{x_1}^{x_2} dx\sqrt{1 + \dot{y}^2} = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2}\,dx \quad \text{— ①}$$

we have the condition for the curve to be shortest minimum is

$$\int_{x_1}^{x_2} f(y, \dot{y}, x)\,dx = 0 \quad \text{— ②}$$

Comparing eq ① and eq ② we have

$$f = \sqrt{1 + \dot{y}^2} \quad \text{— ③}$$

Then

$$\frac{\partial f}{\partial y} = 0$$

and

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}(1 + \dot{y}^2)^{1/2}$$

01-03-06

$$\frac{\partial f}{\partial \dot{y}} = \frac{1}{\cancel{2}}(1 + \dot{y}^2)^{-1/2}(\cancel{2}\dot{y}) = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

In Term of Euler Equation we have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

putting the required values we have

$$\Rightarrow \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = 0$$

$$\text{or} \quad d\,\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = 0 \times dx = 0$$

Integrating we have

$$\int d\,\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \int 0$$

$$\Rightarrow \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{constt} = c \quad \text{— ④}$$

From eq ④ we have

$$\dot{y} = (1 + \dot{y}^2)^{1/2}\,c$$

$$\text{or} \quad \dot{y}^2 = (1 + \dot{y}^2)\,c^2$$

$$\text{or} \quad \dot{y}^2 = c^2 + c^2\dot{y}^2$$

$$\dot{y}^2 - c^2\dot{y}^2 = c^2$$

$$\dot{y}^2(1 - c^2) = c^2$$

$$\text{or} \quad \dot{y}^2 = \frac{c^2}{(1 - c^2)}$$

$$\text{or} \quad \dot{y} = \frac{c}{\sqrt{1 - c^2}} = a \quad \text{— ⑤}$$

01-03-06

where a is another constant As $\dot{y} = \frac{dy}{dx}$

$\Rightarrow \dot{y} = a$

$\frac{dy}{dx} = a$

or $dy = a\,dx$

Integrating we have

$\int dy = a \int dx$

or $\boxed{y = ax + b}$

where b is constant.

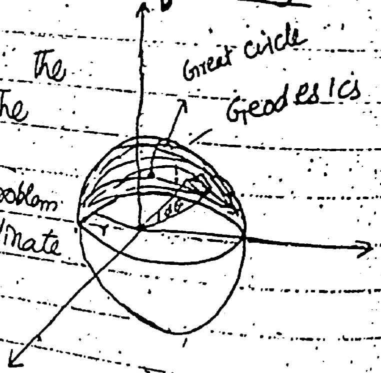
The above equation is equation of straight line with "a" and "b" are constant

Here The condition of extremum is used which is minimum.

* HA *

② Shortest distance between TWO Points on The (sphere) Surface of The sphere :-

The curve which gives the shortest distance on The surface is called Geodesics. In This problem The spherical polar Coordinate are used.



Setting up The arc Length in Term of angle Coordinates of Position on The surface of sphere

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$ds^2 = r^2 [d\theta^2 + \sin^2\theta\, d\phi^2]$$

$$\Rightarrow ds = r[d\theta^2 + \sin^2\theta\, d\phi^2]^{\frac{1}{2}}$$

01-03-06

The total length of the curve is.

$$I = \int_1^2 ds$$

putting The value of ds

$$I = \int_{\theta_1}^{\theta_2} r[d\theta^2 + \sin^2\theta\, d\phi^2]^{1/2}$$

Here r is constant which is radius

$$= \int_{\theta_1}^{\theta_2} r\,d\theta \left[1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2\right]^{1/2}$$

As

$$= \int_{\theta_1}^{\theta_2} r\,d\theta [1 + \sin^2\theta (\dot{\phi})^2]^{\frac{1}{2}} \quad \left(\text{we have } \dot{\phi} = \frac{d\phi}{dt}\right)$$

$$I = \int_{\theta_1}^{\theta_2} r[1 + \sin^2\theta\, \dot{\phi}^2]^{1/2} d\theta \quad \text{—— (A)}$$

But we know That

$$\int_{x_1}^{x_2} f(y, \dot{y}, x)\,dx = 0 \quad \sim (B)$$

In The same form then

$$f = r[1 + \sin^2\theta\, \dot{\phi}^2]^{1/2} \cancel{d\theta} \sim (1)$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial}{\partial \phi}\left[r[1 + \sin^2\theta\, \dot{\phi}^2]^{1/2} \cancel{d\theta}\right]$$

$$\frac{\partial f}{\partial \phi} = 0$$

Diff. eq ① w.r. to $\dot{\phi}$ we have

$$\frac{\partial f}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left[r[1 + \sin^2\theta\, \dot{\phi}^2]^{1/2} \cancel{d\theta}\right]$$

$$= r\left[\frac{\partial}{\partial \dot{\phi}}(1 + \sin^2\theta\, \dot{\phi}^2)^{1/2}\right] \cancel{d\theta}$$

$$= r\left[\frac{1}{2}(1 + \sin^2\theta\, \dot{\phi}^2)^{-\frac{1}{2}} (\sin^2\theta (2\dot{\phi}) \frac{d\dot{\phi}}{d\dot{\phi}}\right]$$

$$\frac{\partial f}{\partial \dot{\phi}} = r\left[\frac{\sin^2\theta\, \dot{\phi}}{\sqrt{(1 + \sin^2\theta\, \dot{\phi}^2)}}\right]$$

01-03-06

As $r \neq 0$ and is constant

$$\frac{\partial f}{\partial \dot{\phi}} = \frac{\sin^2\theta\,\dot{\phi}}{\sqrt{(1+\sin^2\theta\,\dot{\phi}^2)}}$$

As we know that

$$\frac{d}{d\theta}\frac{\partial f}{\partial \dot{\phi}} - \frac{\partial f}{\partial \phi} = 0$$

putting the values we have

$$\therefore \frac{d}{d\theta}\left[\frac{\sin^2\theta\,\dot{\phi}}{\sqrt{1+\sin^2\theta\,\dot{\phi}^2}}\right] = 0$$

$$d\left[\frac{\sin^2\theta\,\dot{\phi}}{\sqrt{1+\sin^2\theta\,\dot{\phi}^2}}\right] = 0 \times d\theta = 0$$

Integrating both sides we have

$$\int d\left(\frac{\sin^2\theta\,\dot{\phi}}{\sqrt{1+\sin^2\theta\,\dot{\phi}^2}}\right) = \int 0$$

$$\frac{\sin^2\theta\,\dot{\phi}}{\sqrt{1+\sin^2\theta\,\dot{\phi}^2}} = c$$

$$\sin^2\theta\,\dot{\phi} = c(1+\sin^2\theta\,\dot{\phi}^2)^{\frac{1}{2}}$$

Squaring both sides we have

$$\sin^4\theta\,\dot{\phi}^2 = c^2(1+\sin^2\theta\,\dot{\phi}^2)$$

$$\sin^4\theta\,\dot{\phi}^2 = c^2 + c^2\sin^2\theta\,\dot{\phi}^2$$

$$\sin^4\theta\,\dot{\phi}^2 - c^2\sin^2\theta\,\dot{\phi}^2 = c^2$$

$$\dot{\phi}^2\sin^2\theta\,(\sin^2\theta - c^2) = c^2$$

$$\dot{\phi}^2 = \frac{c^2}{\sin^2\theta(\sin^2\theta - c^2)}$$

Taking under root on both side

95 (48)

01-03-06

$$\dot{\phi} = \frac{c}{\sin\theta(\sin^2\theta - c^2)^{1/2}}$$

$$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta(\sin^2\theta - c^2)^{1/2}}$$

$$= \frac{c}{\sin\theta\,\sin\theta\left(1-\frac{c^2}{\sin^2\theta}\right)^{1/2}}$$

$$= \frac{c}{\sin^2\theta(1-c^2\operatorname{cosec}^2\theta)^{1/2}}$$ As $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$$= \frac{c}{\sin^2\theta(1-c^2(1+\cot^2\theta))^{1/2}}$$ As $\operatorname{cosec}^2\theta = 1+\cot^2\theta$

$$= \frac{c\operatorname{cosec}^2\theta}{(1-c^2-c^2\cot^2\theta)^{1/2}}$$

$$= \frac{c\operatorname{cosec}^2\theta}{(1-c^2)^{1/2}\left[1-\frac{c^2}{(1-c^2)}\cot^2\theta\right]^{1/2}}$$

$$= \frac{c\operatorname{cosec}^2\theta}{\sqrt{1-c^2}\left(1-\frac{c^2}{(1-c^2)}\cot^2\theta\right)^{1/2}}$$

$$= \frac{c}{\sqrt{1-c^2}}\,\frac{\operatorname{cosec}^2\theta}{\left(1-\frac{c^2}{(1-c^2)}\cot^2\theta\right)^{1/2}}$$

Let $\frac{c}{\sqrt{1-c^2}} = \alpha$ & $\frac{c}{\sqrt{1-c^2}} = k$

then $\Rightarrow k^2 = c$

$$\frac{d\phi}{d\theta} = \frac{\alpha\operatorname{cosec}^2\theta}{(1-\alpha^2\cot^2\theta)^{1/2}}$$

$$d\phi = \frac{\alpha\operatorname{cosec}^2\theta\,d\theta}{\sqrt{(1-\alpha^2\cot^2\theta)}} \quad \text{---} \; ①$$

Let $x = \alpha\cot\theta \Rightarrow dx = -\alpha\operatorname{cosec}^2\theta\,d\theta$

$-dx = \alpha\operatorname{cosec}^2\theta\,d\theta$

06-03-06

putting these values in eq ② we have

$$d\phi = \frac{-dx}{\sqrt{1-x^2}}$$ integrating we get

$$\int d\phi = -\int \frac{dx}{\sqrt{1-x^2}}$$

$$\phi = -\sin^{-1}(x) + \alpha$$

As $x = \alpha \cot\theta$ then

$$\phi = -\sin^{-1}(\alpha\cot\theta) + \alpha$$

$$\text{or} \quad \phi = \alpha - \sin^{-1}(\alpha\cot\theta)$$

$$(+\phi - \alpha) = -\sin^{-1}(\alpha\cot\theta)$$

$$\sin^{-1}(\alpha\cot\theta) = (\alpha - \phi)$$

$$\alpha\cot\theta = \sin(\alpha - \phi)$$

$$\text{or} \quad \frac{\alpha\cos\theta}{\sin\theta} = \sin(\alpha-\phi)$$

$$\text{or} \quad \alpha\cos\theta = \sin(\alpha-\phi)\sin\theta$$

where $\alpha = k$ then

$$k\cos\theta = \sin(\alpha-\phi)\sin\theta$$

Using the relation between cartesian coordinate (x,y,z) and spherical polar coordinate (r,θ,ϕ) we can

$$k\cos\theta = (\sin\alpha\cos\phi - \sin\phi\cos\alpha)\sin\theta$$

$$k\cos\theta = \sin\alpha\cos\phi\sin\theta - \sin\phi\cos\alpha\sin\theta$$

As $\cos\theta = \frac{z}{r}$ putting in above

$$\frac{kz}{r} = \sin\theta\cos\phi\sin\alpha - \sin\theta\sin\phi\cos\alpha$$

$$kz = r\sin\theta\cos\phi\sin\alpha - r\sin\theta\sin\phi\cos\alpha$$

$$\boxed{kz = x\sin\alpha - y\cos\alpha}$$

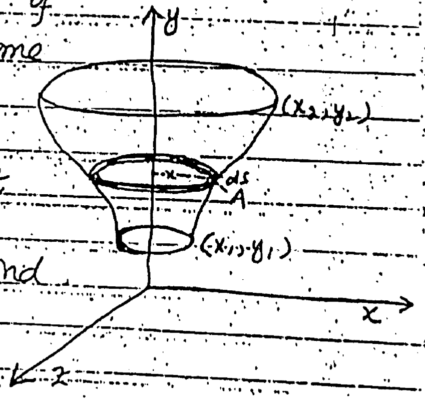
where $r^2 = x^2+y^2+z^2$

where
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$

The last equation represent a plane passing through the origin and hence cutting the surface of the sphere in great circle. Thus the extremum value of the distance between the two points on the surface of the sphere

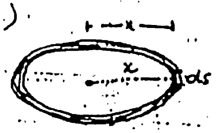
06-03-06

③ MINIMUM SURFACE OF REVOLUTION:-



Suppose we form a surface of revolution by taking some curve passing through two points (x_1, y_1) and (x_2, y_2) and revolving it about y-axis.
Here the problem is to find that curve for which the surface area is minimum.

Let us consider a strip (curve) at a point "A" formed due to revolution of arc length ds.



radius = r = x

If distance of this arc is "x" from the y-axis

Then the differential area of the strip is equal to

$$2\pi x\, ds$$

Since it is xy-plane i.e $ds = \sqrt{dx^2+dy^2}$

Then $dA = (2\pi x\, ds)$

$$= 2\pi x\sqrt{dx^2+dy^2}$$

$$= 2\pi x\, dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

As $\dot{y} = \frac{dy}{dx}$

$$dA = 2\pi x\, dx\sqrt{1+\dot{y}^2}$$

$$dA = 2\pi x\sqrt{1+\dot{y}^2}\, dx \quad \text{—①}$$

The total are of the strip is

$$J = \int_1^2 dA$$

06.03.06

putting the value of dA from eq ①

$$J = \int_{x_1}^{x_2} 2\pi x\sqrt{1+y^2}\,dx \sim Ⓐ$$

As we know that

$$J = \int_{x_1}^{x_2} f\,dx \quad —Ⓑ$$

comparing eq Ⓐ and eq Ⓑ we have

$$f = 2\pi x\sqrt{1+\dot{y}^2}$$

The 2π is constant Then it is removed because it is not equal to zero Then.

$$f = x\sqrt{1+\dot{y}^2}$$

Diff The above eq w.r.t y and $\dot{y}$ we have

$$\frac{\partial f}{\partial y} = 0$$

&

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} x\sqrt{1+\dot{y}^2}$$

$$= \frac{1}{2}x(1+\dot{y}^2)^{-1/2}(2\dot{y})$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{(1+\dot{y}^2)}}$$

The condition that the integral J is an extremum is $\delta J = 0$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

or

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 0$$

i-e The Euler's Equation is zero

06-03-06

putting The values of $\frac{\partial f}{\partial \dot{y}}$ and $\frac{\partial f}{\partial y}$ we have

$$\frac{d}{dx}\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} - 0 = 0$$

$$d\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = 0\times dx = 0$$

Integrating both sides we have

$$\int d\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = \int 0$$

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a = \text{Constt}$$

Squaring both sides we have

$$\frac{x^2\dot{y}^2}{(1+\dot{y}^2)} = a^2$$

By cross multiplication

$$x^2\dot{y}^2 = a^2(1+\dot{y}^2)$$

$$x^2\dot{y}^2 = a^2 + a^2\dot{y}^2$$

$$x^2\dot{y}^2 - a^2\dot{y}^2 = a^2$$

$$\dot{y}^2(x^2-a^2) = a^2$$

$$\dot{y}^2 = \frac{a^2}{(x^2-a^2)},$$

Taking under root on both sides

$$\dot{y} = \frac{a}{(x^2-a^2)^{1/2}}$$

$$\frac{dy}{dx} = \frac{a}{(x^2-a^2)^{1/2}} \qquad \text{As } \dot{y} = \frac{dy}{dx}$$

$$dy = \frac{a}{(x^2-a^2)^{\frac{1}{2}}}\,dx$$

Integrating both sides we have

$$\int dy = a\int \frac{dx}{\sqrt{(x^2-a^2)}}$$

$$y = a\cosh^{-1}\frac{x}{a} + b$$

$$\text{or} \quad y-b = a\cosh^{-1}\frac{x}{a}$$

$$\text{or} \quad \cosh^{-1}\frac{x}{a} = \frac{y-b}{a}$$

$$\text{or} \quad \frac{x}{a} = \cosh\left(\frac{y-b}{a}\right)$$

$$x = a\cosh\left(\frac{y-b}{a}\right)$$

which is the Equation of Catnary.

* *

Yet we have discuss (3) Three application of calculus of variation

① In the first application we have find the minimum Line

② In the second application we see that the surface is minimum

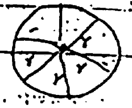
③ In the third application we have discuss that the Area is minimum

④ In the fourth application which will be discussed we will discuss about that the Time is minimum which is called Brachistrochrone Problem.

*

Written By MATI ULLAH BANOCHI
M.Sc (Previous)

Radius:- The distance between the Locus of any curve which is equidistant from a fixed point called the center of the curve is called the Radius of the required curve. The radius of the straight line is infinite.



* *

④ BRACHISTOCHRONE OR

(TARWAL METHOD)

Shottest time Problem

In this problem we find a curve joining the two points along which a Particle falling from rest under the influence of gravity (uniform force field.) Travels from higher to lower point in minimum Time.

We wish to find the path that would be Taken up by the Particle along which its time of Transit between the two points is minimum. Orient the Coordinate System such that the initial point at rest is Taken the origin.

Suppose $\vec{V}$ is the speed of the Particle along the curve, then in Traversing the ds portion of the curve, time spent will be $\frac{ds}{V}$. The total time Taken from point '1' to point '2' will be

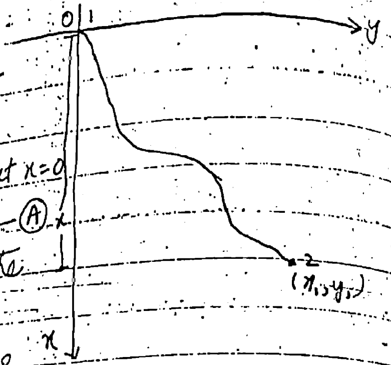
$$t_{12} = \int_1^2 \frac{ds}{V} \quad \text{—} ①$$

If the frictional forces are ignored we can write the total ~~energy~~ of the particle T+V= const. Since the Force field is given Conservative

If the potential Energy is Taken to be Zero at the point $x=0$

$v = -mgx = -mg(0) = 0$ at $x=0$

then we have $T+V=0$ — Ⓐ at the origin, since the particle is at rest there.



Now in any point (x_1,y_1) along the path, the kinetic energy will be $T = K.E = \frac{1}{2}mv^2$ and potential Energy will be $V = -Fx = -mgx$ where g is the acceleration of the particle due to force

putting these values in eq Ⓐ

$$\frac{1}{2}mv^2 - mgx = 0$$

$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gx$$

$$\Rightarrow v^2 = 2gx \Rightarrow v = \sqrt{2gx}$$

Then eq ② will be

$$t_{12} = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{ds}{\sqrt{2gx}} = \int_1^2 \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gx}}$$

$$= \int_0^{x_1} \frac{dx\sqrt{1+(\frac{dy}{dx})^2}}{\sqrt{2gx}} = \int_0^{x_1} \frac{dx\sqrt{1+\dot{y}^2}}{\sqrt{2gx}} \quad \text{As } \frac{dy}{dx} = \dot{y}$$

$$t_{12} = \int_0^{x_1} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gx}}\,dx = \int_0^{x_1} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gx}}\,dx$$

Comparing with

$$t_{12} = \int_{x_1}^{x_1} f\,dx \quad \text{we get}$$

$$f = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gx}}$$

08-03-06

Since the factor $(2g)^{-1/2}$ does not effect the final equation, we can write functional f as

$$f = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{x}} = \left[\frac{1+\dot{y}^2}{x}\right]^{1/2}$$

Then $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1+\dot{y}^2}{x}\right]^{1/2} = 0$

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left[\frac{1+\dot{y}^2}{x}\right]^{1/2}$$

$$= \frac{1}{2}(1+\dot{y}^2)^{-1/2}(2\dot{y})(x)^{-1/2}$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{x}\sqrt{1+\dot{y}^2}} = \frac{\dot{y}}{\sqrt{x(1+\dot{y}^2)}}$$

We have the Eular Equation as

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 0$$

putting the values we have

$$\frac{d}{dx}\frac{\dot{y}}{\sqrt{x(1+\dot{y}^2)}} = 0$$

By multiplying the dx to other side

$$d\frac{\dot{y}}{\sqrt{x(1+\dot{y}^2)}} = 0 \cdot dx = 0$$

let $\Rightarrow \frac{\dot{y}}{\sqrt{x(1+\dot{y}^2)}} = \frac{1}{\sqrt{2a}}$ Note this point

squaring we get

$$\frac{\dot{y}^2}{x(1+\dot{y}^2)} = \frac{1}{2a}$$

$$2a\dot{y}^2 = x(1+\dot{y}^2)$$

$$2a\dot{y}^2 = x + x\dot{y}^2$$

$$2a\dot{y}^2 - x\dot{y}^2 = x$$

$$2ay^2 - xy^2 = x$$

$$y^2(2a - x) = x$$

$$y^2 = \frac{x}{(2a-x)}$$

Taking under root on both sides

$$y = \frac{\sqrt{x}}{\sqrt{(2a-x)}}$$

as $\frac{dy}{dx} = y$ putting this value

$$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{(2a-x)}}$$

$$dy = \frac{\sqrt{x}}{\sqrt{(2a-x)}}\,dx$$

Integrating both sides

$$\int dy = \int \frac{\sqrt{x}}{\sqrt{(2a-x)}}\,dx$$

$$y = \int \frac{\sqrt{x}}{\sqrt{(2a-x)}}\,dx \quad \text{— Ⓑ}$$

Let $\boxed{x = a(1-\cos\theta)}$ — ⓐ

$$dx = a\sin\theta\,d\theta$$

Then

$$\sqrt{\frac{x}{(2a-x)}} = \sqrt{\frac{a(1-\cos\theta)}{2a-(a(1-\cos\theta)}}$$

$$= \sqrt{\frac{\cancel{a}(1-\cos\theta)}{\cancel{a}(\cancel{2}-\cancel{1}+\cos\theta)}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{\cancel{2}\sin^2\theta/2}{\cancel{2}\cos^2\theta/2}} = \sqrt{\frac{\sin^2\theta/2}{\cos^2\theta/2}} = \sqrt{\tan^2\frac{\theta}{2}}$$

105

08-03-06

$$\Rightarrow \sqrt{\frac{x}{(2a-x)}} = \tan\theta/2$$

Putting these values in eq Ⓑ we have

$$y = \int \tan\frac{\theta}{2}\cdot a\sin\theta\,d\theta = a\int\frac{\sin\theta/2}{\cos\theta/2}\,2\sin\theta/2\,\cos\theta/2\,d\theta$$

$$= 2a\int\sin^2\theta/2\,d\theta = \int \cancel{2}a\,\frac{(1-\cos\theta)}{\cancel{2}}\,d\theta = \int a(1-\cos\theta)\,d\theta$$

$$\Rightarrow y = \int a(1-\cos\theta)\,d\theta$$

$$= a\int d\theta - a\int\cos\theta\,d\theta \qquad x = a(1-\cos\theta)$$

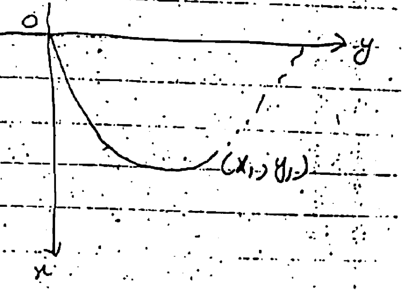
$$y = a(\theta - \sin\theta) + \text{constt} \quad \text{— ⓑ}$$

The constant term is equal to zero initially when $x=0,\ y=0$ and $\theta=0$. Thus we have

$$x = a(1-\cos\theta) \quad \text{from eq ⓐ \& eq ⓑ}$$

$$y = a(\theta - \sin\theta)$$

These are the equations of cycloid passing through the origin. Thus the path of particle is cycloid. The value of constant a must be adjusted such that the particle passes through the other point (x_1, y_1) is found to be minimum although the variational principle is for an extremum value.



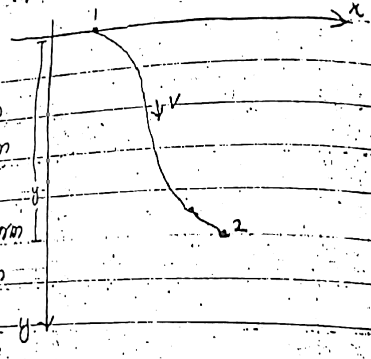
The path of particle is cycloid.

~~~ ☆ ~~~ ☆ ~~~
~~~

08-03-06

"Goldstein Method"

④ BRACHISTOCHRONE OR Shortest Time PROBLEM



In This problem, we find a curve joining two points along which a particle falling from rest under the influence of gravity (uniform force field), travels from higher to a lower point in The minimum time.

Suppose $\vec{v}$ is The speed of The particle along The curve, Then in Traversing ds portion of curve, time spent will be $\frac{ds}{v}$. The total time taken from Point 1 to point 2 will be

$$t_{12} = \int_1^2 \frac{ds}{v} \quad \text{—— ①}$$

Suppose The vertical distance of fall upto point '2' be y. Then from The Conservation of Energy we find

$$K.E = P.E$$

$$\frac{1}{2}mv^2 = mgy \qquad (h=y)$$

$$\Rightarrow v^2 = 2gy$$

$$\Rightarrow v = \sqrt{2gy}$$

Then putting this value in eq ①, we have

08-03-06

$$t_{12} = \int_1^2 \frac{ds}{\sqrt{2gy}}$$

Here $ds = \sqrt{dy^2 + dx^2}$ because it is x y plane

$$= \int_1^2 \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}}$$

As $\dot{y} = \frac{dy}{dx}$

$$= \int_1^2 \frac{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} = \int_1^2 \frac{dx\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$$

$$t_{12} = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\,dx$$

Hence The function will be $f = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$

As $\sqrt{2g}$ is constant Hence

$$f = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}} = \sqrt{1+\dot{y}^2}\,(y)^{-1/2} \quad \text{—— Ⓐ}$$

then

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{1+\dot{y}^2}\,(y)^{-1/2}\right)$$

$$= -\frac{1}{2}\left(\sqrt{1+\dot{y}^2}\; y^{-3/2}\right) \quad \text{—— ⓐ}$$

and diff eq Ⓐ w.r.to $\dot{y}$ we have

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(\sqrt{1+\dot{y}^2}\,(y)^{-1/2}\right)$$

$$= \frac{1}{2}(1+\dot{y}^2)^{-1/2}(\cancel{2}\dot{y})(y)^{-1/2}$$

$$\frac{\partial f}{\partial \dot{y}} = \dot{y}(1+\dot{y}^2)^{-1/2}(y)^{-1/2} \quad \text{—— ⓑ}$$

The condition for which t_{12} will be extremum (minimum) when

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

08-03-08

putting the values from ⓐ + ⓑ we have

$$-\frac{1}{2}(1+\dot{y}^2)^{\frac{1}{2}}y^{-\frac{3}{2}}-\frac{d}{dx}\left[\dot{y}(1+\dot{y}^2)^{-\frac{1}{2}}y^{-\frac{1}{2}}\right]=0$$

$$-\frac{1}{2}(1+\dot{y}^2)^{\frac{1}{2}}y^{-\frac{3}{2}}-\left[\frac{d}{dx}\left[\dot{y}(1+\dot{y}^2)^{-\frac{1}{2}}y^{-\frac{1}{2}}\right]\right]=0$$

$$-\frac{1}{2}(1+\dot{y}^2)^{\frac{1}{2}}y^{-\frac{3}{2}}-\left[-\frac{1}{2}y^{-\frac{3}{2}}\dot{y}\dot{y}(1+\dot{y}^2)^{-\frac{1}{2}}-\frac{1}{2}(1+\dot{y}^2)^{-\frac{3}{2}}(2\dot{y})(\ddot{y})\dot{y}y^{-\frac{1}{2}}\right.$$

$$\left.+\ddot{y}(1+\dot{y}^2)^{-\frac{1}{2}}y^{-\frac{1}{2}}\right]=0$$

$$-\frac{1}{2}(1+\dot{y}^2)y^{-\frac{3}{2}}+\frac{1}{2}y^{-\frac{3}{2}}\dot{y}^2(1+\dot{y}^2)^{-\frac{1}{2}}+\frac{1}{2}(1+\dot{y}^2)^{-\frac{3}{2}}\cancel{2}\dot{y}^2\ddot{y}y^{-\frac{1}{2}}$$

$$-\ddot{y}(1+\dot{y}^2)^{-\frac{1}{2}}y^{\frac{1}{2}}=0$$

After simplyfying we get
Taking common from all Term we get

$$(1+\dot{y}^2)^{-3/2}y^{-3}\quad[1+\dot{y}^2+2y\ddot{y}]=0$$

Either one or other will be equal to zero To make it wholy zero.

Either $1+\dot{y}^2+2y\ddot{y}=0$

This equation is differential Equation the Problem of finding The shape of the path is called Brachistrochrone

As $1+\dot{y}^2+2y\ddot{y}=0$ — Ⓑ

Let $\dot{y}=u$

then $\ddot{y}=\frac{du}{dx}=\frac{du}{dy}\left(\frac{dy}{dx}\right)$ (As $\frac{dy}{dx}=\dot{y}$)

$$\ddot{y}=\dot{y}\frac{du}{dy}=u\frac{du}{dy}$$

putting These values in eq Ⓑ we have

08-03-08

$$1+u^2+2yu\frac{du}{dy}=0$$

or $\frac{1}{2yu}\frac{dy}{du}+\frac{1}{1+u^2}=0$

or $\frac{dy}{y}+\frac{2u}{1+u^2}du=0$

Integrating both sides we have

$$\int\frac{dy}{y}+\int\frac{2u}{(1+u^2)}du=\int 0$$

$$\ln y+\ln(1+u^2)=\ln b$$

where b is a constant

$$\ln y(1+u^2)=\ln b$$

$$y(1+u^2)=b$$

or $y+u^2y=b$

or $u^2y=b-y$

or $u^2=\frac{b-y}{y}$

Taking under root we get $\Rightarrow u=\sqrt{\frac{b-y}{y}}$

As $u=\dot{y}=\frac{dy}{dx}$ putting its value in above equation we get

$$\frac{dy}{dx}=\sqrt{\frac{b-y}{y}}$$

or $\Rightarrow \frac{dx}{dy}=\sqrt{\frac{y}{(b-y)}}$

$$dx=\sqrt{\frac{y}{(b-y)}}$$

Taking integration on both sides we get

$$\int dx=\int\sqrt{\frac{y}{(b-y)}}\,dy \quad —\text{Ⓒ}$$

08-03-06

$x = \int \sqrt{\frac{y}{b-y}}\, dy + c$ ~ Ⓜ

let $y = b\sin^2\theta$ ~ ②

Then $dy = 2b\sin\theta\cos\theta\, d\theta$

putting This values in above equation we have

$$x = \int \sqrt{\frac{b\sin^2\theta}{(b - b\sin^2\theta)}}\,(2b\sin\theta\cos\theta\, d\theta)$$

$$= 2b \int \sqrt{\frac{b\sin^2\theta}{b(1-\sin^2\theta)}}\,(\sin\theta\cos\theta\, d\theta)$$

$$= 2b \int \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\,(\sin\theta\cos\theta\, d\theta)$$

$$= 2b \int \sqrt{\tan^2\theta}\,(\sin\theta\cos\theta\, d\theta)$$

$$= 2b \int \tan\theta\,(\sin\theta\cos\theta\, d\theta)$$

$$= 2b \int \frac{\sin\theta}{\cancel{\cos\theta}}\,(\sin\theta\,\cancel{\cos\theta}\, d\theta)$$

$$= 2b \int \sin^2\theta\, d\theta \qquad \left(\text{As } \sin^2\theta = \frac{1-\cos2\theta}{2}\right)$$

$$= 2b \int \left(\frac{1-\cos2\theta}{2}\right) d\theta$$

$$= \frac{2b}{2}\left[\int d\theta - \int \cos2\theta\, d\theta\right]$$

$$= b\left[\theta - \frac{\sin2\theta}{2}\right] + c$$

$$x = \frac{1}{2}b\,[2\theta - \sin2\theta] + c \quad ③$$

08-03-06

Thus The parametric equations of required curve are which we obtained from eq ② & eq ③ we have

$$x = \frac{1}{2}b\,(2\theta - \sin2\theta) + c \quad ©$$

&

$$y = b\sin^2\theta = \frac{b}{2}\,(1-\cos2\theta)$$

or

$$y = \frac{1}{2}b\,(1-\cos2\theta) \quad Ⓓ$$

$\left(\text{where } \sin^2\theta = \frac{1-\cos2\theta}{2}\right)$

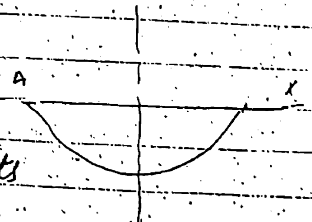
The curve pass through from $x = 0$, $y = 0$ putting in eq Ⓜ we get $c = 0$

Then let $\phi = 2\theta$ and $a = \frac{1}{2}b$

putting This values in eq © & eq Ⓓ we get

$$\boxed{x = a(\phi - \sin\phi)}$$
$$\boxed{y = a(1-\cos\phi)}$$

These are The Parametric Equations of cycloid.



The shape of cyclocid is such That Curve drawn by a Point on The circle (circle) as its rolls down is called cyclocid.

* * *

HOME ASSIGNMENT

Show That Geodesics of a cylender is a Helix.

Cylendrical coord. (ρ, θ, z)

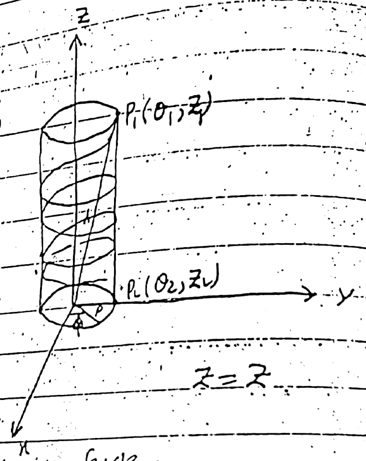
Solve it Necessary For Exam Point of View.

$ds = \sqrt{\rho^2 d\phi^2 + dz^2}$

The assignment is solved on the next page.

13-03-06

⑤ SHOW THAT GEODESICS OF A CYLENDER IS A HELIX :-



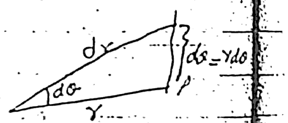
The curved or spring type path inside the cylender is called helical path or spiral path.
The curved surface is a function of r and z. We will solve this question by the cylendrical coordinate system.

$z = z$

The total area segment will be

$$ds = \sqrt{d\theta^2 + dz^2}$$

Then its representation (As in cylendrical coordinates will be



$$ds = \sqrt{r^2 d\theta^2 + dz^2} \qquad \because d\theta = r d\alpha$$

θ is changing and the z is also changing when the body is coming down through the helical path

$$ds = dz \sqrt{\frac{r^2 d\theta^2}{dz^2} + 1}$$

$$\text{or} \quad ds = dz \sqrt{1 + r^2 \left(\frac{d\alpha}{dz}\right)^2}$$

Then we have $\qquad \because \dot{\theta} = \frac{d\alpha}{dz}$

$$ds = dz \sqrt{1 + r^2 \dot{\theta}^2}$$

13-03-06

Then the total distance from P_1 to P_2 will be given that

$$J = \int_1^2 ds = \int_1^2 dz (1 + r^2 (\dot{\theta})^2)^{1/2}$$

$$J = \int_1^2 (1 + r^2 (\dot{\theta})^2)^{1/2} dz \quad \text{——} ①$$

Then the J is minimum when

$$\frac{\partial f}{\partial \theta} - \frac{d}{dz} \frac{\partial f}{\partial \dot{\theta}} = 0 \quad \text{——} Ⓐ$$

From eq ① we have

$$f = (1 + r^2 (\dot{\theta})^2)^{1/2}$$

Diff w.r.t θ we have

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} (1 + r^2 (\dot{\theta})^2)^{1/2} = 0$$

$$\frac{\partial f}{\partial \theta} = 0$$

Then

$$\frac{\partial f}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} (1 + r^2 (\dot{\theta})^2)^{1/2}$$

$$= \frac{1}{2} (1 + r^2 (\dot{\theta})^2)^{-1/2} (2 \dot{\theta} r^2)$$

$$\frac{\partial f}{\partial \dot{\theta}} = \frac{\dot{\theta} r^2}{(1 + r^2 \dot{\theta}^2)^{1/2}}$$

putting these values in eq Ⓐ we have

$$0 - \frac{d}{dz} \left(\frac{\dot{\theta} r^2}{(1 + r^2 \dot{\theta}^2)^{1/2}} \right) = 0$$

$$\text{or} \quad \frac{d}{dz} \left(\frac{\dot{\theta} r^2}{(1 + r^2 \dot{\theta}^2)^{1/2}} \right) = 0$$

$$d \left(\frac{r^2 \dot{\theta}}{(1 + r^2 \dot{\theta}^2)^{1/2}} \right) = 0 \times dz = 0$$

Integrating both sides we have

$$\int d\frac{(r^2\dot\theta)}{(1+r^2\dot\theta^2)^{1/2}} = \int 0$$

$$\frac{r^2\dot\theta}{(1+r^2\dot\theta^2)^{1/2}} = C_1$$ where C_1 is Constant

By cross multiplication

$$r^2\dot\theta = C_1(1+r^2\dot\theta^2)^{1/2}$$

Squaring both sides we have

$$r^4\dot\theta^2 = C_1^2(1+r^2\dot\theta^2)$$

$$r^4\dot\theta^2 = C_1^2 + C_1^2 r^2\dot\theta^2$$

$$r^4\dot\theta^2 - C_1^2 r^2\dot\theta^2 = C_1^2$$

$$r^2\dot\theta^2(r^2 - C_1^2) = C_1^2 \qquad C_1^2 = C_1^2$$

$$r^2\dot\theta^2 = \frac{C_1^2}{(r^2-C_1^2)}$$

Taking under root on both sides

$$r\dot\theta = \frac{C_1}{\sqrt{(r^2-C_1^2)}} = C_2$$

$$r\dot\theta = C_2 \qquad \text{Taking } \frac{C_1}{\sqrt{r^2-C_1^2}} = C_2$$

$$r\frac{d\theta}{dz} = C_2 \qquad \text{As } \dot\theta = \frac{d\theta}{dz} \text{ then}$$

$$r\,d\theta = C_2\,dz$$

$$d\theta = \frac{C_2}{r}dz$$

Integrating both sides we have

$$\int d\theta = \frac{C_2}{r}\int dz$$

$$\boxed{\theta = \frac{C_2}{r}z + C_3}$$

where C_3 is another constant

115

13-03-06

In some book it is also written that

$$\theta = \frac{a}{r}z + b$$

where "a" and "b" are constants then

$$\boxed{\theta = \frac{C_2}{r}z + C_3}$$

This Equation is the parametric equation of the helix

* * *

Some Applications of calculus of variation are left which we will do in the last inshaullah.

* * *

Hamilton' EQUATION OF MOTION:-

FOR Full details see #331 Takwall

We have to look equivalent formulation to Newtonian mechanics

In the Lagrangian mechanics we did not consider the forces but we consider only the energies which was kinetic energy and Potential Energy.

Our Lagrangian was a function of q_j, $\dot q_j$ and t. but the Lagrangian formulation ended up with second order differential equation in time

$$\vec F = \dot{\vec P} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = 0$$

(for conservative system)

where $L = T - V$

Both of the above are second order differential equation.

14-03-06

In Hamilton formulations, the coordinates are transformed from $(q_j, \dot{q}_j, t)$ to the (q_j, P_j, t) where q_j and P_j are two canonical independent generalized coordinates ∴

$$H = H(q_j, P_j, t)$$

This will end up with a first order differential equation.

Now for deriving the Hamilton equation of motion see Takwall Book at page 311

* * *

LEGENDER TRANSFORMATION

Consider a function

$$f = f(x_1, x_2 \dots x_n, y_1, y_2 \cdots y_n, t) \text{ —— } ①$$

The set of $\{x_i\}$ is called active variable and $\{y_i\}$ is called passive variables then we have

$$\left.\begin{aligned} \frac{\partial f}{\partial x_i} &= u_i \\ \frac{\partial f}{\partial y_i} &= v_i \end{aligned}\right\} \text{ —— } ②$$

where $i = 1, 2, 3 \dots$

Differentating eq ① we have

$$df = \sum_i \frac{\partial f_i}{\partial x_i} dx_i + \sum_i \frac{\partial f}{\partial y_i} dy_i + \frac{\partial f}{\partial t} dt$$

Putting the values from eq ② we have

$$df = \sum_i u_i dx_i + \sum_i v_i dy_i + \frac{\partial f}{\partial t} dt \text{ — } ③$$

14-03-06

Let us define the legendre Transformation

$$g = \sum_i u_i x_i - f \text{ — } ④$$

Then

$$dg = d\left[\sum_i u_i x_i - f\right]$$

$$dg = \sum \left[u_i dx_i + x_i du_i\right] - df$$

Putting the value of df in above equation from the eq ③

$$dg = \sum \left[u_i dx_i + x_i du_i\right] - \sum u_i dx_i - \sum v_i dy_i - \frac{\partial f}{\partial t} dt$$

$$= \sum u_i dx_i + \sum x_i du_i - \sum u_i dx_i - \sum v_i dy_i - \frac{\partial f}{\partial t} dt$$

$$dg = \sum_i (x_i du_i - v_i dy_i) - \frac{\partial f}{\partial t} dt \text{ — } ⑤$$

* By comparing eq ⑤ & eq ③ we have

$$\text{generally } \left.\frac{\partial g}{\partial u_i} = x_i \quad \frac{\partial g}{\partial y_i} = -v_i \; ; \; \frac{\partial g}{\partial t} = \frac{-\partial f}{\partial t}\right\} \text{ — } ⑥$$

This means that $g = g(u_i, y_i, t)$

The two are like converse ② & ⑥

$$\because \quad g = g(u_i, y_i, t)$$

$$g = g(u_1, u_2 \dots u_n, y_1, y_2, y_3 \cdots y_n, t)$$

$$g = g(u_i, y_i, t)$$

then

$$dg = \sum \frac{\partial g}{\partial u_i} du_i + \sum \frac{\partial g}{\partial y_i} dy_i + \frac{\partial g}{\partial t} dt \text{ — } ⑦$$

By comparing eq ⑤ & eq ⑦ we get

$$x_i = \frac{\partial g}{\partial u_i}, \qquad \frac{\partial g}{\partial t} = -\frac{\partial f}{\partial t}$$

$$-v_i = \frac{\partial g}{\partial y_i}$$

* * *

15.03.06

Hamilton' EQUATION OF MOTION :-

The Lagrangian function L is given as
$$L = L(\dot{q}_j, q_j, t)$$

then
$$dL = \sum_i \frac{\partial L}{\partial q_j} dq_j + \sum_i \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \quad ①$$

Since as we know that
$$\frac{\partial L}{\partial \dot{q}_j} = P_j$$ which is conjugate or General momentum.

Then As $\frac{\partial L}{\partial \dot{q}_j} = P_j$

$$\frac{\partial L}{\frac{d}{dt}(\partial q_j)} = P_j \Rightarrow \frac{\partial L}{\partial q_j} = \frac{d}{dt} P_j = \dot{\vec{P}}_j$$

$$\Rightarrow \frac{\partial L}{\partial q_j} = \dot{\vec{P}}$$

As $\frac{\partial L}{\partial q_j} = \dot{\vec{P}}_j$ & $\frac{\partial L}{\partial \dot{q}_j} = \vec{P}_j$

putting these values in eq ① we have

$$dL = \sum_i \dot{\vec{P}}_j dq_j + \sum_i \vec{P}_j d\dot{q}_j + \frac{\partial L}{\partial t} dt \quad ②$$

Let us consider
$$\sum d(P_j \dot{q}_j) = \sum P_j d\dot{q}_j + \sum \dot{q}_j dP_j$$

$$\sum P_j d\dot{q}_j = \sum d(P_j \dot{q}_j) - \sum \dot{q}_j dP_j$$

putting The value of $P_j d\dot{q}_j$ in eq ② we have

$$dL = \sum_i \dot{\vec{P}}_j dq_j + \sum d(P_j \dot{q}_j) - \sum \dot{q}_j dP_j + \frac{\partial L}{\partial t} dt$$

$$dL - \sum d(P_j \dot{q}_j) = \sum_i \dot{\vec{P}} dq_j - \sum \dot{q}_j dP_j + \frac{\partial L}{\partial t} dt$$

119

15.03.06

or $-d\sum[-L + P_j \dot{q}_j] = -[\sum(-\dot{q}_j dP_j - \dot{P}_j dq_j) - \frac{\partial L}{\partial t} dt]$

or $d[-L + P_j \dot{q}_j] = [\sum(\dot{q}_j dP_j - \dot{P}_j dq_j) - \frac{\partial L}{\partial t} dt]$

As $H = \sum \dot{q}_j P_j - L$ which is Hamiltonian

Then $dH = d(\sum \dot{q}_j P_j - L)$

putting this value in above equation

$$dH = \sum(\dot{q}_j dP_j - \dot{P}_j dq_j) - \frac{\partial L}{\partial t} dt \quad ③$$

As we know That Hamiltonian equation is a function of
$$H = H(q_j, P_j, t)$$

Then
$$dH = \sum_i \left(\frac{\partial H}{\partial q_j} dq_j\right) + \sum_i \frac{\partial H}{\partial P_j} dP_j + \frac{\partial H}{\partial t} dt \quad ④$$

By comparing eq ③ and eq ④ we have

$$\left.\frac{\partial H}{\partial q_j} = -\dot{P}_j, \quad \frac{\partial H}{\partial P_j} = \dot{q}_j, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \right\} \quad ⑤$$

The above eq ⑤ is called Hamilton equation of motion or Hamilton Canonical equation of motion. They form a set of 2n first order differential equations of motion and replace the n-lagrange equations of second order.

So for n generalized coordinates we have 2n number of first order differential equation where
$$H = T + V$$

15.03.06

These equations have replaced n-second order differential equation in Lagrangian formulation by 2n first order differential equation in Hamiltonian.

ONE DIMENSIONAL CASE :-

Here we consider only one dimensional case and consider the one dimension as the x- direction

The Hamiltonian is

$$H = T + V \quad \text{—— (A)}$$

The kinetic energy will be

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{\dot{x}}^2 \quad \text{—— (1)}$$

And the potential energy is

$$V = V(x)$$

Putting the value of T & V in eq (A)

$$H = T + V = \frac{1}{2} m \vec{\dot{x}}^2 + V(x)$$

$$H = \frac{1}{2} m \vec{\dot{x}}^2 + V(x) \quad \text{—— (B)}$$

$$H = H(x, P_x)$$

Now Take $T = \frac{1}{2} m \vec{\dot{x}}^2$

diff w.r.t $\dot{x}$ we have

$$\frac{\partial T}{\partial \dot{x}} = \frac{1}{2} m \dot{x}^{2-1} (2) = m\dot{x}$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = m\dot{x} = P_x$$

Then eq (B) will be written as

$$H = \frac{1}{2} m \vec{\dot{x}}^2 + V(x)$$

or

$$H = \frac{1}{2m} m^2 \vec{\dot{x}}^2 + V(x)$$

15.03.06

As $P_x = m\dot{x}$

Then $P_x^2 = m^2 \vec{\dot{x}}^2$

Then

$$H = \frac{m^2 \vec{\dot{x}}^2}{2m} + V(x)$$

$$H = \frac{P_x^2}{2m} + V(x) \quad \text{—— (C)}$$

From this equation we have

Diff it w.r.t x we have

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x} V(x)$$

$$\frac{\partial H}{\partial x} = + \frac{\partial V(x)}{\partial x} = -\vec{\dot{P}}_x$$

or

$$\boxed{\vec{\dot{P}}_x = -\frac{\partial V}{\partial x} = \vec{F}_x}$$

Or differentiating eq (C) w.r.t P_x we have

$$\frac{\partial H}{\partial P_x} = \frac{\partial}{\partial P_x} \left(\frac{P_x^2}{2m} + V(x) \right)$$

(As $P = mV$, $V_x = \dot{x}$)

$$= \frac{\cancel{2} P_x}{\cancel{2} m} = \frac{P_x}{m} = \frac{\cancel{m} V_x}{\cancel{m}}$$

or

$$\frac{\partial H}{\partial P_x} = V_x = \vec{\dot{x}}$$

$$\Rightarrow \frac{P_x}{m} = \vec{\dot{x}}$$

$$\Rightarrow \boxed{P_x = m\dot{x} = (mV)_x}$$

The General Case in cartesian coordinate will be

$$H = \frac{1}{2m} (P_x^2 + P_y^2 + P_z^2) + V(x, y, z)$$

20-03-06

Hamilton's Equations in A cartesian coordinate system :-

The Hamiltonian in cartesian coordinates are normally given as $H = T + V$

$$H = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + V_{(x,y,z)} \quad \text{——} Ⓐ$$

This is the general equation of the Hamiltonian in the Cartesian coordinate system. Then it have six components. Differentiating eq Ⓐ w.r.t x, y, z Then

$$\frac{\partial H}{\partial x} = \frac{\partial V}{\partial x} = -\dot{P}_x \Rightarrow \boxed{\dot{P}_x = -\frac{\partial V}{\partial x}}$$

$$\frac{\partial H}{\partial y} = \frac{\partial V}{\partial y} = -\dot{P}_y \Rightarrow \boxed{\dot{P}_y = -\frac{\partial V}{\partial y}}$$

$$\frac{\partial H}{\partial z} = \frac{\partial V}{\partial z} = -\dot{P}_z \Rightarrow \boxed{\dot{P}_z = -\frac{\partial V}{\partial z}}$$

then Differentiating eq Ⓐ w.r.t P_x, P_y & P_z respectively we get

$$\frac{\partial H}{\partial P_x} = \frac{2P_x}{2m} = \frac{P_x}{m} = \dot{x} \Rightarrow \boxed{P_x = m\dot{x}}$$

$$\frac{\partial H}{\partial P_y} = \frac{2P_y}{2m} = \frac{P_y}{m} = \dot{y} \Rightarrow \boxed{P_y = m\dot{y}}$$

$$\frac{\partial H}{\partial P_z} = \frac{2P_z}{2m} = \frac{P_z}{m} = \dot{z} \Rightarrow \boxed{P_z = m\dot{z}}$$

Thus the Hamiltonian in the Cartesian Coordinate will be

$$H = \sum \dot{q}_j P_j - L$$

$\Rightarrow$

which gives us 20-03-06

$$H_{(x,y,z)} = (\dot{x}P_x + \dot{y}P_y + \dot{z}P_z) - L$$

where

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

* * *

The Lagrangian equation is given as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

In this case the order of the equation is 'n' and it is 2nd order differential equation. But in case of Hamiltonian we have

$$\frac{\partial H}{\partial q_j} = -\dot{P}_j \,,\ \frac{\partial H}{\partial P_j} = \dot{q}_j$$

This equation have n-coordinates and we have 2n number of 1st order differential equations.

* * *

HAMILTONIAN EQUATIONS IN PLANE POLAR COORDINATES (r, θ) :-

As we know that the Hamiltonian is given as $H = \sum_j \dot{q}_j P_j - L$ In the plane polar coordinates the Hamiltonian will be given as

$$H_{(r,\theta)} = \dot{r}P_r + \dot{\theta}P_\theta - L \quad \text{——} ①$$

∴ Here we have

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V_{(r,\theta)}$$

then from the above equation as

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

20-03-06

$$T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) \quad \text{— (A)}$$

Then from eq (A)

$$\frac{\partial L}{\partial \dot r} = \frac{\partial T}{\partial \dot r} = \frac{1}{2}m\,\cancel{2}\dot r = m\dot r = P_r \Rightarrow P_r = m\dot r \Rightarrow \boxed{\dot r = \frac{P_r}{m}}$$

$$\frac{\partial L}{\partial \dot\theta} = \frac{\partial T}{\partial \dot\theta} = \frac{1}{\cancel{2}}m(\cancel{2}r^2\dot\theta) = mr^2\dot\theta = P_\theta \Rightarrow \boxed{\dot\theta = \frac{P_\theta}{mr^2}}$$

~~(A) we have $\dot r = \frac{P_r}{m}$ and $\dot\theta = \frac{P_\theta}{mr^2}$ putting these values in eq (1) we have~~

$$H_{(r,\theta)} = \dot r P_r + \dot\theta P_\theta - L$$

$$H_{(r,\theta)} = \dot r P_r + \dot\theta P_\theta - \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + V_{(r,\theta)}$$

The above equation is obtained by putting the value of L in eq (1)

$$H_{(r,\theta)} = \dot r P_r + \dot\theta P_\theta - \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + V_{(r,\theta)} \quad \text{— (2)}$$

Rewriting the above equation in term of generalized coordinates and also putting the value of $\dot r$ and $\dot\theta$ in it

$$H_{(r,\theta)} = \frac{P_r}{m}(P_r) + \frac{P_\theta}{mr^2}(P_\theta) - \frac{1}{2}m\left(\frac{P_r^2}{m^2} + \frac{r^2P_\theta^2}{m^2r^4}\right) + V_{(r,\theta)}$$

$$= \frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} - \frac{1}{2}\frac{\cancel{m}P_r^2}{m^2} - \frac{1}{2}\frac{\cancel{m}r^2P_\theta^2}{m^2r^4} + V_{(r,\theta)}$$

$$= \frac{P_r^2}{m} - \frac{1}{2}\frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} - \frac{P_\theta^2}{2mr^2} + V_{(r,\theta)}$$

$$= \frac{2P_r^2 - P_r^2}{2m} + \frac{2P_\theta^2 - P_\theta^2}{2mr^2} + V_{(r,\theta)}$$

$$H_{(r,\theta)} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V_{(r,\theta)} \quad \text{— (3)}$$

This is Hamiltonian in the plane polar coordinates

20-03-06

Differentiating eq (3) with respect to r we have

$$\frac{\partial H}{\partial r} = \frac{\partial}{\partial r}\left[\frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V_{(r,\theta)}\right] = \frac{-2P_\theta^2}{2mr^3} + \frac{\partial V}{\partial r} = -\dot P_r$$

$$\Rightarrow -\frac{P_\theta^2}{mr^3} + \frac{\partial V}{\partial r} = -\dot P_r$$

or

$$\boxed{\dot P_r = \frac{P_\theta^2}{mr^3} - \frac{\partial V}{\partial r}}$$

This is called the Radial force

Similarly diff eq (3) w.r.t θ we get

$$\frac{\partial H}{\partial \theta} = \frac{\partial V}{\partial \theta} = -\dot P_\theta \Rightarrow \boxed{\dot P_\theta = -\frac{\partial V}{\partial \theta}}$$

Similarly for differentiating for P_r and P_θ we have

$$\frac{\partial H}{\partial P_r} = \frac{2P_r}{2m} = \dot r \Rightarrow \dot r = \frac{P_r}{m} = \boxed{P_r = m\dot r}$$

This is called the radial part

Then for P_θ we have

$$\frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2} = \dot\theta \Rightarrow \boxed{P_\theta = mr^2\dot\theta}$$

This specifies the angular part

Hence as a whole the Hamiltonian equation of motions in plane polar coordinates will be given as

$$\dot P_r = \frac{P_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot P_\theta = -\frac{\partial V}{\partial \theta}$$

$$P_r = m\dot r$$

$$P_\theta = mr^2\dot\theta$$

MATIULLAH - KHAN
ON 20 March 2006
22 years of Completion
SALGARA MUBARIK
Cut The CAKE FOR
20 MARCH 1984

*

21-03-06

HAMILTON'S EQUATION FOR FRE PARTICLE:

In case of free particle we have potential Energy is equal to zero i.e $V=0$ The total energy or The Hamiltonian is

$$H = T+V = E$$

As $V=0$ then

$$H = E = T+0 = T$$

~~But $T = \frac{1}{2}mv^2$~~

~~$H = E = T = \frac{1}{2}mv^2$~~

But if we consider only x-direction Then

$$H = \frac{1}{2}m\dot{x}^2 \qquad \text{As } P_x = m\dot{x}$$

$$= \frac{1}{2m}m^2\dot{x}^2 \qquad P_x^2 = m^2\dot{x}^2$$

$$H = \frac{P_x^2}{2m}$$

The Schrodinger Equation is given as

$$H\psi = E\psi$$

putting values of H we have

$$\left(\frac{P_x^2}{2m}\right)\psi - E\psi = 0$$

$$\text{or} \quad \left(\frac{P_x^2}{2m} - E\right)\psi = 0 \quad \text{—— ①}$$

As we know That momentum in schrodinger case is

$$\bar{P} = -i\hbar\nabla$$

$$\text{or} \quad \bar{P}_x = -i\hbar\frac{d}{dx}$$

$$\text{or} \quad P_x = \frac{-i^2}{i}\hbar\frac{d}{dx} = \frac{\hbar}{i}\frac{d}{dx}$$

21-03-06

putting The value of P_x in eq ① we have

$$\left(\frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 - E\right)\psi = 0$$

$$\text{or} \quad \left(+\frac{1}{2m}\hbar^2\frac{d^2}{dx^2} - E\right)\psi = 0$$

$$\text{or} \quad \left(\frac{d^2}{dx^2} - \frac{2mE}{\hbar^2}\right)\psi = 0$$

$$\text{or} \quad \frac{d^2}{dx^2}\psi - \frac{2mE}{\hbar^2}\psi = 0$$

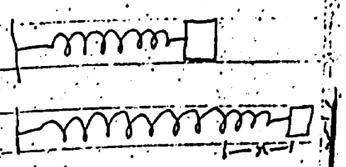
It is equation for The free Particle

* * *

EQUIVALENCY BETWEEN THE NEWTONIAN, LAGRANGAIN AND HAMILTONIAN

The equivalency of These three can be found from The example of mass attach spring in which all the (equations) Formulations gives The same equation.

Consider a mass which is attached with a Spring we apply The Force on it to displace it from The mean position.



NEWTONIAN WAY:-

We know That

$$F = ma \quad \text{—①}$$

But the restoring force when The spring attach mass is displaced from the mean position

$$F = -kx \quad \text{—②}$$

Comparing eq ① + eq ② we have

$ma = -kx$ But $a = \frac{d^2x}{dt^2} = \ddot{x}$

Then $a = \frac{-k}{m}x$

or $\frac{d^2x}{dt^2} = \frac{-k}{m}x$

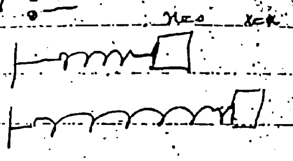
or $\ddot{x} = \frac{-k}{m}x$

$\ddot{x} + \frac{k}{m}x = 0$ where $\omega = \sqrt{\frac{k}{m}}$

$\boxed{\ddot{x} + \omega^2 x = 0}$

This is the required Equation for the mass attached spring (spring attach mass) by Newtonian way.

LAGRANGIAN WAY:-



In Lagrangian we have

$L = T - V$ —①

Here $T = \frac{1}{2}m\dot{x}^2$

But $V = F_{av} \cdot x$ where $F_{average} = \frac{0 + kx}{2} = \frac{1}{2}kx$

Then $V = \frac{1}{2}kx \cdot x = \frac{1}{2}kx^2$

putting values of T & V in eq ①

$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ —Ⓐ

From eq Ⓐ diff w.r. $\dot{x}$ we have

$\frac{\partial L}{\partial \dot{x}} = \frac{2m\dot{x}}{2} = m\dot{x}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x}$

Similarly Diff eq Ⓐ w.r.t x we have

$\frac{\partial L}{\partial x} = -\frac{2kx}{2} = -kx$

The Lagrangian Equation of motion is

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$ putting the values we get

$m\ddot{x} - (-kx) = 0$

$m\ddot{x} + kx = 0$

$\ddot{x} + \frac{k}{m}x = 0$

or $\boxed{\ddot{x} + \omega^2 x = 0}$ where $\omega = \sqrt{\frac{k}{m}}$

The same equation as obtained from Newtonian way

HAMILTON'S WAY:-

In Hamiltonian we have

$H = \Sigma \dot{q}_j p_j - L$ $H = T + V$

$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx$

But in special case here

$q_j = x$ and $p_j = p_x$ Then

$H = \dot{x}p_x - L$

As we know that $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

putting in above we have

$H = \dot{x}p_x - (\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2)$

$H = \dot{x}p_x - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

putting $\dot{x} = \frac{p_x}{m}$ in above

But

$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x$

$\Rightarrow p_x = m\dot{x}$

$\therefore \dot{x} = \frac{p_x}{m}$

$H = \left(\frac{p_x}{m}\right)p_x - \frac{1}{2}m\left(\frac{p_x}{m}\right)^2 + \frac{1}{2}kx^2$

$= \frac{p_x^2}{m} - \frac{1}{2}m\frac{p_x^2}{m^2} + \frac{1}{2}kx^2$

$= \frac{2p_x^2 - p_x^2}{2m} + \frac{1}{2}kx^2$

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 \quad \text{— (A)}$$

This is the Hamiltonian. Now we are going to find out the Hamiltonian equations of motion for the spring attached mass.

As $\frac{\partial H}{\partial q_j} = -\dot{p}_x$, $\frac{\partial H}{\partial p_j} = \dot{q}_j$

But here $q_j = x$ & $\dot{q}_j = \dot{x}$. Then diff eq (A) with respect to x we have

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left(\frac{p_x^2}{2m}\right) + \frac{1}{2}k(2x) = 0 + kx$$

or $\frac{\partial H}{\partial x} = kx = -\dot{p}_x$ As $\dot{p}_x = F_x$

$\Rightarrow \dot{p}_x = -kx = F_x$

$\Rightarrow \dot{p}_x = F_x = -kx$ — ①

Similarly differentiating eq (A) w.r. to p_x we h

$$\frac{\partial H}{\partial p_x} = \frac{2p_x}{2m} = \frac{p_x}{m} = \dot{x}$$

$\Rightarrow p_x = m\dot{x} = m\dot{x}$

diff w.r.t t we have

$$\frac{\partial p_x}{\partial t} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

$$\frac{\partial p_x}{\partial t} = \dot{p}_x = m\ddot{x}$$

or $\dot{p}_x = m\ddot{x}$ — ②

Comparing eq ① and eq ② we have

$\dot{p}_x = m\ddot{x} = -kx$

or $m\ddot{x} + kx = 0$

$\ddot{x} + \frac{k}{m}x = 0$

Which is required equation or $\boxed{\ddot{x} + \omega^2 x = 0}$ where $\omega = \sqrt{\frac{k}{m}}$

HOME ASSIGNMENT

27-03-06

(a) Write Hamiltonian in

(1) spherical polar coordinates (r, θ, ϕ)

(2) cylendrical coordinates (ρ, ϕ, z)

① HAMILTON'S EQUATIONS IN Spherical polar Coordinates :-

Since we know that

$$H(r,\theta,\phi) = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L \quad \text{— ①}$$

and

$$H = \sum_j \dot{q}_j p_j - L$$

The $L = T - V = \frac{1}{2}mV^2 - V(r,\theta,\phi)$ — ⓐ

But

$$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$$

Then eq ⓐ ⇒

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V(r,\theta,\phi) \quad \text{— (A)}$$

Then putting the value of L in eq ① we have

$$H(r,\theta,\phi) = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) + V(r,\theta,\phi) \quad \text{— ②}$$

Since we know that

$$\frac{\partial L}{\partial \dot{q}_j} = p_j$$

so

eq (A) $\Rightarrow \frac{\partial L}{\partial \dot{r}} = m\dot{r} = p_r \Rightarrow \boxed{\dot{r} = \frac{p_r}{m}}$

eq (A) $\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = p_\theta \Rightarrow \boxed{\dot{\theta} = \frac{p_\theta}{mr^2}}$

eq (A) $\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}mr^2\sin^2\theta = p_\phi \Rightarrow \boxed{\dot{\phi} = \frac{p_\phi}{r^2 m \sin^2\theta}}$

Putting the values of $\dot{r}$, $\dot{\theta}$ & $\dot{\phi}$ in eq ② we get

$$H_{(r,\theta,\phi)} = \left(\frac{p_r}{m}\right)p_r + \left(\frac{p_\theta}{mr^2}\right)p_\theta + \left(\frac{p_\phi}{r^2 m \sin^2\theta}\right)p_\phi$$

$$-\frac{1}{2}\left(\frac{p_r^2}{\cancel{m^2}}\cancel{m} + \frac{p_\theta^2}{m^2r^2}m + \frac{p_\phi^2}{r^2 m \sin^2\theta}\right) + V_{(r,\theta,\phi)}$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{r^2 m \sin\theta} - \frac{1}{2}\left(\frac{p_r^2}{m}\right) - \frac{p_\theta^2}{2mr^2} - \frac{p_\phi^2}{2r^2 m \sin^2\theta} + V_{(r,\theta,\phi)}$$

$$H_{(r,\theta,\phi)} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2r^2 m \sin^2\theta} + V_{(r,\theta,\phi)}$$

Hence required Equation of Hamilton in spherical coordinate is

$$\boxed{H_{(r,\theta,\phi)} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2r^2 m \sin^2\theta} + V_{(r,\theta,\phi)}}$$

* *

HAMILTON'S EQUATION IN CYLENDRICAL Coordinates :-

Since we know that

$$H = \sum \dot{q}_j p_j - L$$

Then

$$H_{(\rho,\phi,z)} = \dot{\rho} p_\rho + \dot{\phi} p_\phi + \dot{z} p_z - L \quad — ①$$

But $L = T - V = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V_{(\rho,\phi,z)}$ — Ⓐ

putting value of L in eq ① we have

$$H_{(\rho,\phi,z)} = \dot{\rho} p_\rho + \dot{\phi} p_\phi + \dot{z} p_z - \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + V_{(\rho,\phi,z)} \quad — ②$$

From eq Ⓐ we have

$$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} = p_\rho \Rightarrow \boxed{\dot{\rho} = \frac{p_\rho}{m}}$$

eq Ⓐ ⇒ $$\frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = p_\phi \Rightarrow \boxed{\dot{\phi} = \frac{p_\phi}{\rho^2 m}}$$

eq Ⓐ ⇒ $$\frac{\partial L}{\partial \dot{z}} = m\dot{z} = p_z \Rightarrow \boxed{\dot{z} = \frac{p_z}{m}}$$

putting these values in eq ① we have

$$H_{(\rho,\phi,z)} = \frac{p_\rho^2}{m} + \frac{p_\phi^2}{\rho^2 m} + \frac{p_z^2}{m} - \frac{1}{2}\left(\frac{p_\rho^2}{m} + \frac{p_\phi^2}{\rho^2 m} + \frac{p_z^2}{m}\right) + V_{(\rho,\phi,z)}$$

$$= \frac{p_\rho^2}{m} + \frac{p_\phi^2}{\rho^2 m} + \frac{p_z^2}{m} - \frac{p_\rho^2}{2m} - \frac{p_\phi^2}{2\rho^2 m} - \frac{p_z^2}{2m} + V_{(\rho,\phi,z)}$$

$$\boxed{H_{(\rho,\phi,z)} = \frac{p_\rho^2}{2m} + \frac{p_\phi^2}{2\rho^2 m} + \frac{p_z^2}{2m} + V_{(\rho,\phi,z)}}$$

which is required Hamilton's Equation in the Cylenderical Coordinates.

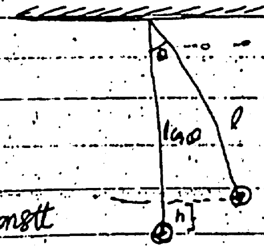
* *

SOME APPLICATIONS OF THE HAMILTONIAN FORMULATION :-

#313

(a) SIMPLE PENDULUM :-

The Hamiltonian and the Hamiltonian Equations for the Simple pendulum will be



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

putt $r = l =$ constt $\Rightarrow \dot{r} = 0$

$$T = \frac{1}{2}m(0 + r^2\dot{\theta}^2)$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

As $l - l\cos\theta = h$

and $V = mg(l - l\cos\theta) = mgh$

$$V = mgh$$

Reference of l is taken below the point of suspension.

$H = T+V = \frac{1}{2} m l^2 \dot{\theta}^2 + mg(l - l\cos\theta)$ — ①

$L = T-V = \frac{1}{2} m l^2 \dot{\theta}^2 - mg(l - l\cos\theta)$ — ②

As $\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} = p_\theta$ By definition

$\Rightarrow \dot{\theta} = \frac{p_\theta}{m l^2}$

putting the value of $\dot{\theta}$ in eq ① we have

$H = \frac{1}{2} m l^2 \left(\frac{p_\theta^2}{m^2 l^4}\right) + mg(l - l\cos\theta)$

$H = \frac{p_\theta^2}{2 m l^2} + mg(l - l\cos\theta)$ — ③

or $H = \sum \dot{q}_j p_j - L$

$H = \dot{\theta} p_\theta - L$

But $L = \frac{1}{2} m l^2 \dot{\theta} - mg(l - l\cos\theta)$

putting value of L in above equation

$H = \dot{\theta} p_\theta - \left(\frac{1}{2} m l^2 \dot{\theta} - mg(l - l\cos\theta)\right)$

$\therefore H = \dot{\theta} p_\theta - \frac{1}{2} m l^2 \dot{\theta} + mg(l - l\cos\theta)$

But $\dot{\theta} = \frac{p_\theta}{m l^2}$ Then above eq will be

$H = \frac{p_\theta^2}{m l^2} - \frac{1}{2} m l^2 \frac{p_\theta^2}{m^2 l^4} + mg(l - l\cos\theta)$

$= \frac{p_\theta^2}{m l^2} - \frac{p_\theta^2}{2 m l^2} + mg(l - l\cos\theta)$

$H = \frac{p_\theta^2}{2 m l^2} + mg(l - l\cos\theta)$ — ③

Now The Hamiltonian equations will be from eq ③ we have

$\frac{\partial H}{\partial q_j} = -\dot{p}_j \Rightarrow \frac{\partial H}{\partial \theta} = mgl\sin\theta = -\dot{p}_\theta$ — Ⓐ

Similarly $\frac{\partial H}{\partial p_j} = \dot{q}_j$

Then from eq ③ we have

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l^2}$

$\Rightarrow p_\theta = m l^2 \dot{\theta}$ — Ⓑ

differentiating above

$\dot{p}_\theta = m l^2 \ddot{\theta}$ — Ⓒ

Comparing eq Ⓐ and eq Ⓒ we have

$m l^2 \ddot{\theta} = -mgl\sin\theta$

or $l^2 \ddot{\theta} + gl\sin\theta = 0$

Dividing by l we have

$l\ddot{\theta} + g\sin\theta$

If $\theta <<$ small Then $\sin\theta \approx \theta$ Then

$l\ddot{\theta} + g\theta = 0$

$\ddot{\theta} + \frac{g}{l}\theta = 0$

or $\ddot{\theta} + \omega^2\theta = 0$ where $\omega = \sqrt{\frac{g}{l}}$

$\Rightarrow \boxed{\ddot{\theta} + \omega^2\theta = 0}$

(b) CHARGE PARTICLE IN AN ELECTROMEGNETIC FIELD:-

The Lagrangain of charged Particle in an electromegnetic field is

$L = T - V = \frac{1}{2} mV^2 - q\phi + q(V \cdot A)$ — ①

$\frac{\partial L}{\partial \dot{q}_k} = p_k \Rightarrow \frac{\partial L}{\partial V_k} = mV_k + qA = P$

or Generally $mV_k + qA_i = P_k$ — (2)

we can also write eq ② as

$$m\vec{V} + q\vec{A} = \vec{P}$$

$$\sum_k \hat{e}_k (mV_k + qA_k) = m\vec{V} + q\vec{A} = \vec{P}$$

The Hamiltonian Function H can be written as

$$H = \sum_j \dot{q}_j P_j - L = \sum_k \dot{q}_k P_k - L$$ putting values of $\dot{q}_k$ + P_k we have

$$H = \sum_k V_k (mV_k + qA_k) - \frac{1}{2}mv^2 + q\phi - \bar{q}(\bar{V}.\bar{A})$$

$$H = \sum_k (mV_k^2 + (V_k A_k) q) - \frac{1}{2}mV^2 + q\phi - \bar{q}(\bar{V}.\bar{A})$$

$$\left(As \sum_k (V_k . A_k) q = q(V_x A_x + V_y A_y + V_z A_z) = q(V.A)\right)$$

$$H = \sum_k (mV_k^2 - \frac{1}{2}mv^2 + q\phi)$$

$$H = mv^2 - \frac{1}{2}mv^2 + q\phi$$

or $$H = \frac{1}{2}mv^2 + q\phi \quad — ③$$

From eq ② we have

$$\bar{P} = m\bar{V} + q\bar{A} \Rightarrow m\bar{V} = P - q\bar{A}$$

or $$mV = P - qA \quad — ④$$

where

$$\underset{\text{net momentum}}{\underset{\downarrow}{P}} = \underset{\text{mechanical momentum}}{\underset{\downarrow}{mV}} + q\bar{A} \text{ (magnetic moment)}$$

then eq ② can be written as

$$H = \frac{1}{2m}(P - q\bar{A})^2 + q\phi$$

or $$H = \frac{1}{2m}(P - q\bar{A})^2 + q\phi \quad — ⑤$$

Now The Hamiltonian Equations will be

$$\frac{\partial H}{\partial q_j} = \dot{P}_j \Rightarrow \frac{\partial H}{\partial q_j} + \dot{P}_j = 0$$

$$\therefore \frac{\partial H}{\partial P_j} = \dot{q}_j \Rightarrow \frac{\partial H}{\partial P_j} - \dot{q}_j = 0$$

$$\frac{\partial H}{\partial P} = \frac{1}{m}(P - qA) = \frac{1}{m}(\bar{P} - q\bar{A}) = \sum_k \dot{V}_k = V$$

$$\frac{\partial H}{\partial q_j} = PH = -\dot{P}_k = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = -\dot{P}_{(x,y,z)}$$

$$\dot{\vec{P}}_k = -\nabla H = -q\nabla\phi + q\nabla(\bar{V}.\bar{A})$$

From eq ⑤ we have

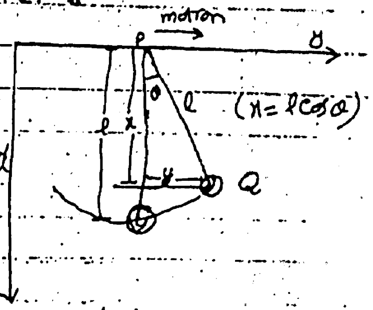
$$H = \frac{1}{2m}(P - q\bar{A})^2 + q\phi$$

and $$H = \frac{P^2}{2m} + \frac{q^2A^2}{2m} - \frac{2PqA}{2m} + q\phi$$

But $$P = mV + qA$$

WRITE HAMILTONIAN FOR A SIMPLE PENDULUM WITH MOVING SUPPORT :-

#313 Takwali



A Pendulum of mass 'm' is suspended from a support which is constrained to move along a straight horizontal line the pendulum oscilates in the vertical plane containing the direction of motion of the support

Let PQ = l.

Then the K.E will be

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad — Ⓐ$$

The Transformation equations are

28-03-06

$x = l\cos\theta$

$Y = y + l\sin\theta$

Since the pendulum had moved y distance the pendulum is moving uniformly along the y-axis.

As $x = l\cos\theta$ $\quad Y = y + l\sin\theta$

$\dot{x} = l(-\sin\theta)\dot{\theta}$ $\quad \dot{Y} = \dot{y} + l\cos\theta\,\dot{\theta}$

or $\dot{x} = -l\sin\theta\,\dot{\theta}$ $\quad \dot{Y} = \dot{y} + l\cos\theta\,\dot{\theta}$

putting these values in eq Ⓐ we have

$$T = \frac{1}{2}m\left[(-l\sin\theta\,\dot{\theta})^2 + (\dot{y} + l\cos\theta\,\dot{\theta})^2\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2\sin^2\theta + \dot{y}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{y}l\dot{\theta}\cos\theta\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + \dot{y}^2 + 2\dot{y}l\dot{\theta}\cos\theta\right]$$

$$T = \frac{1}{2}m\left[\dot{y}^2 + l^2\dot{\theta}^2 + 2\dot{y}\dot{\theta}l\cos\theta\right] \quad \text{— ①}$$

The Potential Energy will be

$V = mg(l - l\cos\theta)$

$V = mg(l - l\cos\theta)$ $\quad$ $V = 0$ is taken at $\theta = 0$

The Lagrangain will be

$$L = T - V = \frac{1}{2}m\left[\dot{y}^2 + l^2\dot{\theta}^2 + 2\dot{y}\dot{\theta}l\cos\theta\right] - mg(l - l\cos\theta)$$

or $$L = \frac{1}{2}m\left[\dot{y}^2 + l^2\dot{\theta}^2 + 2\dot{y}\dot{\theta}l\cos\theta\right] - mg(l - l\cos\theta) \quad ②$$

Since the Hamiltonian is

$$H = \sum_j p_j\dot{q}_j - L$$

Here it will be

$$H = p_y\dot{y} + p_\theta\dot{\theta} - L \quad \text{—} ③$$

28-03-06

In order to find $\dot{y}$ & $\dot{\theta}$ goto eq ② we have

we know that

$$\frac{\partial L}{\partial \dot{q}_j} = p_j$$

eq ② $\Rightarrow$ $$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + ml\dot{\theta}\cos\theta = p_y$$

or $$p_y = m\dot{y} + ml\dot{\theta}\cos\theta \quad \text{—} ⓐ$$

eq ② $\Rightarrow$ $$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + m\dot{y}l\cos\theta = p_\theta$$

or $$p_\theta = ml^2\dot{\theta} + m\dot{y}l\cos\theta \quad \text{—} (b)$$

multiplying eq ⓐ by $l\cos\theta$ and substracting from eq ⓑ we get

$$p_y l\cos\theta = m\dot{y}l\cos\theta + ml^2\dot{\theta}\cos^2\theta$$

$$\underline{\pm p_\theta = \pm m\dot{y}l\cos\theta \pm ml^2\dot{\theta}}$$

or $$p_y l\cos\theta - p_\theta = ml^2\dot{\theta}\cos^2\theta - ml^2\dot{\theta}$$

or $$ml^2\dot{\theta}(\cos^2\theta - 1) = (p_y l\cos\theta - p_\theta)$$

or $$ml^2\dot{\theta}(-\sin^2\theta) = (p_y l\cos\theta - p_\theta)$$

or $$l\dot{\theta} = \frac{p_y l\cos\theta}{-ml\sin^2\theta} - \frac{p_\theta}{ml\sin^2\theta}$$

or $$\boxed{l\dot{\theta} = \frac{-1}{m\sin^2\theta}\left(p_y\cos\theta - \frac{p_\theta}{l}\right)} \quad \text{—} Ⓐ$$

Similarly multiplying eq ⓐ by l and eq ⓑ by $\cos\theta$

$$p_y l = m\dot{y}l + ml^2\dot{\theta}\cos\theta$$

$$\underline{\pm p_\theta\cos\theta = m\dot{y}l\cos^2\theta \pm ml^2\dot{\theta}\cos\theta}$$

$$(p_y l - p_\theta\cos\theta) = m\dot{y}l - m\dot{y}l\cos\theta$$

$$(p_y l - p_\theta\cos\theta) = \dot{y}ml(1 - \cos^2\theta)$$

$$(p_y l - p_\theta\cos\theta) = \dot{y}ml(\sin^2\theta)$$

$$\boxed{\dot y = \frac{1}{m\sin^2\theta}\left(P_y - \frac{P_\theta}{l}\cos\theta\right)} \quad \text{— (B)}$$

squaring eq (B) we have

$$\dot y^2 = \frac{1}{m^2\sin^4\theta}\left(P_y^2 + \frac{P_\theta^2}{l^2}\cos^2\theta - \frac{2P_yP_\theta}{l}\cos\theta\right)$$

$$\frac{1}{2}m\dot y^2 = \frac{\cancel{m}}{2m^2\sin^4\theta}\left(P_y^2 + \frac{P_\theta^2}{l^2}\cos^2\theta - \frac{2P_yP_\theta}{l}\cos\theta\right)$$

$$\frac{1}{2}m\dot y^2 = \frac{P_y^2}{2m\sin^4\theta} + \frac{P_\theta^2\cos^2\theta}{2l^2m\sin^4\theta} - \frac{P_yP_\theta\cos\theta}{ml\sin^4\theta} \quad \text{— (4)}$$

Similarly squaring eq (A) we have

$$l^2\dot\theta^2 = \frac{1}{m^2\sin^4\theta}\left(P_y^2\cos^2\theta + \frac{P_\theta^2}{l^2} - \frac{2P_yP_\theta}{l}\cos\theta\right)$$

$$\frac{1}{2}ml^2\dot\theta^2 = \frac{P_y^2\cos^2\theta}{2m\sin^4\theta} + \frac{P_\theta^2}{2l^2m\sin^4\theta} - \frac{P_yP_\theta\cos\theta}{ml\sin^4\theta} \quad \text{— (5)}$$

Similarly multiplying eq (A) & eq (B) we get

$$\dot y l\dot\theta = \frac{1}{m\sin^2\theta}\left(P_y - \frac{P_\theta}{l}\cos\theta\right)\times\left(\frac{1}{m\sin^2\theta}\left(P_y\cos\theta - \frac{P_\theta}{l}\right)\right)$$

$$= \frac{1}{m^2\sin^4\theta}\left(P_y^2\cos\theta - \frac{P_yP_\theta}{l}\cos^2\theta - \frac{P_yP_\theta}{l} + \frac{P_\theta^2}{l^2}\cos\theta\right)$$

$$\dot y l\dot\theta = -\frac{P_y^2\cos\theta}{m^2\sin^4\theta} + \frac{P_yP_\theta\cos^2\theta}{m^2l\sin^4\theta} + \frac{P_yP_\theta}{lm^2\sin^4\theta} - \frac{P_\theta^2\cos\theta}{m^2l^2\sin^4\theta}$$

$$m\dot y l\dot\theta\cos\theta = -\frac{P_y^2\cos^2\theta}{m\sin^4\theta} + \frac{P_yP_\theta\cos^3\theta}{ml\sin^4\theta} + \frac{P_yP_\theta\cos\theta}{ml\sin^4\theta} - \frac{P_\theta^2\cos^2\theta}{ml^2\sin^4\theta} \quad \text{(6)}$$

As we know that Hamiltonian is

$$H = \sum_k P_k\dot q_k - L = P_y\dot y + P_\theta\dot\theta - L$$

putting the values of P_y, P_θ & L

we have

$$H = m(\dot y^2 + \dot y l\dot\theta\cos\theta) + m(l^2\dot\theta^2 + \dot y l\dot\theta\cos\theta) - \frac{1}{2}m\dot y^2 - \frac{1}{2}ml^2\dot\theta^2 - \dot y l\dot\theta m\cos\theta + mgl(1-\cos\theta)$$

(11)

29-03-06

$$H = m\dot y^2 - \frac{1}{2}m\dot y^2 + m\dot y l\dot\theta\cos\theta + ml^2\dot\theta^2 - \frac{1}{2}ml^2\dot\theta^2 - \dot y l\dot\theta\cancel{m}\cos\theta + m\cancel{\dot y l\dot\theta}\cos\theta + mgl(1-\cos\theta)$$

$$H = \frac{1}{2}m\dot y^2 + \frac{1}{2}ml^2\dot\theta^2 + m\dot y l\dot\theta\cos\theta + mgl(1-\cos\theta)$$

putting the values from eq (4), (5) & (6) we get

$$H = \frac{P_y^2\cos^2\theta}{2m\sin^4\theta} + \frac{P_\theta^2}{2l^2m\sin^4\theta} - \frac{P_yP_\theta\cos\theta}{ml\sin^4\theta} + \frac{P_y^2}{2m\sin^4\theta} + \frac{P_\theta^2\cos^2\theta}{2l^2m\sin^4\theta} - \frac{P_yP_\theta\cancel{\cos\theta}}{\cancel{ml}\sin^4\theta} - \frac{P_y^2\cos^2\theta}{m\sin^4\theta} + \frac{P_yP_\theta\cos^3\theta}{ml\sin^4\theta} + \frac{P_yP_\theta\cancel{\cos\theta}}{\cancel{ml}\sin^4\theta} - \frac{P_\theta^2\cos^2\theta}{ml^2\sin^4\theta} + mgl(1-\cos\theta)$$

$$H = -\frac{P_y^2\cos^2\theta}{2m\sin^4\theta} + \frac{P_y^2}{2m\sin^4\theta} + \frac{P_\theta^2}{2l^2m\sin^4\theta} - \frac{P_\theta^2\cos^2\theta}{2ml^2\sin^4\theta} + \frac{P_yP_\theta\cos^3\theta}{ml\sin^4\theta} - \frac{P_yP_\theta\cos\theta}{ml\sin^4\theta} + mgl(1-\cos\theta)$$

$$= \frac{P_y^2}{2m\sin^4\theta}(1-\cos^2\theta) + \frac{P_\theta^2}{2l^2m\sin^4\theta}(1-\cos^2\theta) + \frac{P_yP_\theta\cos\theta}{ml\sin^4\theta}(\cos^2\theta-1) + mgl(1-\cos\theta)$$

$$= \frac{P_y^2}{2m\sin^{\cancel{4}}\theta}\cancel{\sin^2\theta} + \frac{P_\theta^2}{2l^2m\sin^{\cancel{4}}\theta}\cancel{\sin^2\theta} + \frac{P_yP_\theta\cos\theta}{ml\sin^{\cancel{4}}\theta}(-\cancel{\sin^2\theta}) + mgl(1-\cos\theta)$$

$$H = \frac{P_y^2}{2m\sin^2\theta} + \frac{P_\theta^2}{2l^2m\sin^2\theta} - \frac{P_yP_\theta\cos\theta}{ml\sin^2\theta} + mgl(1-\cos\theta)$$

$$\boxed{H = \frac{1}{2m\sin^2\theta}\left(P_y^2 + \frac{P_\theta^2}{l^2} - \frac{2P_yP_\theta\cos\theta}{l}\right) + mgl(1-\cos\theta)}$$

Since y is cyclic, the momentum P_y is a constant of motion.

28-03-06

Cyclic OR IGNORABLE Coordinates.

Consider

$L = L(q_1, q_2 \dots q_n, \dot{q}_1, \dot{q}_2 \dots \dot{q}_n, t)$

or $L = L(q_j, \dot{q}_j, t)$

If L is such that L is not a function of q_j

Then from $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$

As $L \neq L(q_j)$

Then $\frac{\partial L}{\partial q_j} = 0$ putt in above

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - 0 = 0$$

or $d\frac{\partial L}{\partial \dot{q}_j} = 0 \times dt$

Integrating we get

$$\int d\frac{\partial L}{\partial \dot{q}_j} = \int 0$$

$$\frac{\partial L}{\partial \dot{q}_j} = \text{Constt} = P_j$$

or $\frac{\partial L}{\partial \dot{q}_j} = P_j$

This is the definition.

In this definition if some coordinate is absent or missing from the Lagrangain function is called ignorable or cyclic coordinate whenever a coordinate is cyclic

28-03-06

Then the conjugate to it is constant of motion. If H does not involve a particular coordinate, say q_k, Then

$$\frac{\partial H}{\partial q_k} = -\dot{P}_k$$

(If H is not a function of q_k $H \neq H(q_k)$

$$\frac{\partial H}{\partial q_k} = -\dot{P}_k = 0$$

or $-\dot{P}_k = 0$

$\dot{P}_k = 0$ integrating

$\Rightarrow P_k = \text{Constt}$

such a coordinate q_k is called the cyclic or ignorable coordinate.

The cyclic coordinate present in the Lagrangain will be absent in the Hamiltonian.

$$L = L(q_j, \dot{q}_j, t)$$

Since $\frac{\partial L}{\partial \dot{q}_j} = P_j$

$\dot{q}_j$ in Lagrangain is a cyclic coordinate

Then

$$H = H(q_j, P_j, t)$$

$$H = H(q_j, \frac{\partial L}{\partial \dot{q}_j}, t)$$

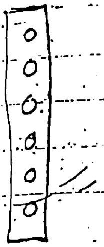
* *

$\boxed{\text{HOME ASSIGNMENT}}$:

WRITE HAMILTONIAN FOR THE COMPOUND PENDULUM.

29-03-06

HAMILTONIAN FOR COMPOUND PENDULUM:-



Consider a compound pendulum whole length is l and the kinetic energy and potential energy 'V' given as

$$T = \frac{1}{2} I \dot{\theta}^2$$

where I is the moment of the inertia.

$$V = -mgl\cos\theta$$

The Lagrangain will be

$$L = T - V$$

$$= \frac{1}{2} I\dot{\theta}^2 - (-mgl\cos\theta)$$

$$L = \frac{1}{2} I\dot{\theta}^2 + mgl\cos\theta$$

As we have $\frac{\partial L}{\partial \dot{\theta}} = P_\theta = \frac{1}{2} I (2\dot{\theta}) = I\dot{\theta}$

$$\Rightarrow P_\theta = I\dot{\theta}$$

The Hamiltonian will be given as

$$H = \sum P_j \dot{q}_j - L$$

$$H = P_\theta \dot{\theta} - L$$

putting the values we have

$$H = (I\dot{\theta})\dot{\theta} - \frac{1}{2} I\dot{\theta}^2 - mgl\cos\theta$$

$$H = I\dot{\theta}^2 - \frac{1}{2} I\dot{\theta}^2 - mgl\cos\theta$$

or $H = \frac{1}{2} I\dot{\theta}^2 - mgl\cos\theta$

or

$$H = \frac{1}{2I} I^2\dot{\theta}^2 - mgl\cos\theta$$

$$H = \frac{(I\dot{\theta})^2}{2I} - mgl\cos\theta \qquad \text{As } I\dot{\theta} = P_\theta$$

$$H = \frac{P_\theta^2}{2I} - mgl\cos\theta$$

which is required equation for compound Pendulum

Physical SIGNIFICANCE OF HAMILTONIAN:-

29-03-06

In Lagrangain Formulation the Lagrangain L is a function of independent variable q_j, the generalized coordinate and its time derivative $\dot{q}_j$. Hence they are not independent variables, for a system of n-degree of freedom, motion is conveniently considered as an n-dimensional coordinate space called configuration space. Since the Lagrangain equation are second order differential equation, we need 2n initial quantities to obtain its solutions.

In the Hamiltonian formulation the picture is however different. In this case the generalized momenta P_j and the generalized coordinate q_j are the independent variables. Thus the velocities are elevated to the role independent variable in the Hamiltonian Formulation and the system with n-degree of freedom has 2n independent variables. The space in which the motion of the system is to be considered is the space decided by the 2n independent variables and is called as phase space. There are now 2n Hamiltonian equations each one being a first order differential equation and their solution will need 2n initial quantities. It is easier to deal with the first order differential equations the ease obtained in solving is however, lost in the number of equations.

The second feature of Hamiltonian equation is that the equation fall in two groups corresponding in generalised momenta and coordinate having an almost symmetrica relationship....

while obtaining the Hamiltonian equation as can be observed from the illstration given above one has to form usually the Lagrangain function for the system, obtain the generalized momenta and then write the Hamiltonian for the system Often it is true that Lagrangain equation can be obtained more easily then Hamiltonian equation Further from the standpoint of the theory of the differential equations, they are identical system and differ only in their form the advantage of Hamiltonian approache is not so much in simplifying the solutions of mechanical problem but it provides the base from which one makes extension to other fields this point can be understood from the fact that the product of generalized momentum and generalized coordinate $p_i q_i$ has always the dimension of action. i.e of energy multiplied by ~~time~~ time this is true no matter what choice of momenta and coordinate is made for mechanical or non mechanical system this is not so in the Lagrangain formulation where as the product $\dot{q}_i q_i$ will always depends on the particular generalised coordinate used

~~30-03-08.~~

Thus the Hamiltonian formulation is particularly useful in making transformation from classical mechanics to quntum mechanics in which the action is quantized.

* * *

OR see on # 32 END OF THIS CHAPTER. Goldstein

HOME ASSIGNMENT Page 340 #

PROBLEM:- show that in thermodynamics the internal Energy $E(S,V)$ Free Energy $P(V,T)$ GIBBS function $G(P,T)$ and the entalphy $H(S,P)$ are related with each other through the Lagender Tansformation

$$\boxed{\text{Hints}} \Longrightarrow \boxed{\begin{aligned} H &= \sum \dot{q}_i p_i - L \\ G &= \sum_i \mu_i x_i - F \end{aligned}}$$

SOLUTION:

Generally we consider a function which has only two variables $f(x,y)$

$$f = f(x,y)$$

then the differential of f can be written as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$df = u\,dx + v\,dy \quad \text{①} \qquad \text{where} \left\{ \begin{aligned} u &= \frac{\partial f}{\partial x} \\ v &= \frac{\partial f}{\partial y} \end{aligned} \right.$$

This above work we done for the general case when the function was dependent on x and y Now we are going to change the quantities on which the function depends. For example the variable x and y here we change by u and y so the differential quantities differential 'du' and 'dy' are expressed in terms of

Let g be a function of 'u' and 'y' and defined by

$g = f - ux$

differentiating above we have

$dg = df - d(ux) = df - udx - xdu$

putting value of df from eq ①

$dg = udx + vdy - udx - xdu$

$dg = vdy - xdu$ — ②

which is exactly the form desired as the quantities x and v are now function of the variables u and y given by

$g = g(u, y)$

$dg = +\frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy$ — ③

Comparing ② & ③ we get $v = \frac{\partial g}{\partial y}, \; x = \frac{\partial g}{\partial u}$

The legender Transformation is frequently used in the thermodynamics For exemple Internal Energy E is a function of E(S, V) then

$E = E(S, V)$

we take $dE = \frac{\partial E}{\partial S}dS + \frac{\partial E}{\partial V}dV$

$\frac{\partial E}{\partial S} = P$ then $\boxed{dE = -Pds + f dV}$

$\frac{\partial E}{\partial V} = S$

Similarly Free energy will be given as $P = P(V, T)$

$\frac{\partial P}{\partial V} = S$ $\quad dP = \frac{\partial P}{\partial V}dV + \frac{\partial P}{\partial T}dT$

$\frac{\partial P}{\partial T} = U$

$\boxed{dP = Sdv + udT}$

The Enthalpy H(S, P) which is function of entropy S and pressure P with properties

where $\frac{\partial H}{\partial S} = T, \; \frac{\partial H}{\partial P} = V$ $\quad H = H(S, P)$

$dH = \frac{\partial H}{\partial S}ds + \frac{\partial H}{\partial p}dp = Tds + Vdp$

$\boxed{dH = Tds + Vdp}$

The GIBBS function may be

$G = X - TS$

$dG = dX - Tds + SdT$

but $dX = Tds + vdp$

$dG = Tds + vdp - Tds - SdT$

$dG = -SdT + vdp$

$G = G(P, T)$

$dG = \frac{\partial G}{\partial p}dp + \frac{\partial G}{\partial T}dT$

$dG = Vdp - SdT$

~~$G = X$~~ $-TS$

~~$dG = -Sd$~~

149

04-April 2006

Motion in a Central Force Field

CHAPTER NO. 5 Book of R.G. TAKWALE # 133

~~CH~~APTER NO. 3 Goldstein Book # 70

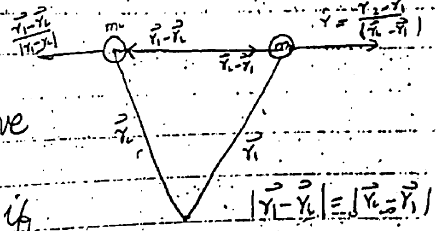
INTRODUCTION :-

MOTION IN A CENTRAL FORCE FIELD OR MOTION IN A CENTRAL POTENTIAL :-

A Central force is that force which acts along the position vector of the two particles drawn from the center of the force. Then it can be expressed as

$$\vec{F}(r) = \hat{r}\,|F(r)| \quad — ①$$

where $\hat{r}$ is the unit vector which is drawn along the direction of the position vector



$\frac{\vec r_1 - \vec r_2}{|\vec r_1 - \vec r_2|}$
m_1
$\vec r_1 - \vec r_2$
$\vec r_2 - \vec r_1$
m_2
$\hat r = \frac{\vec r_2 - \vec r_1}{|\vec r_2 - \vec r_1|}$
$\vec r_2$
$\vec r_1$
$|\vec r_1 - \vec r_2| = |\vec r_2 - \vec r_1|$

If F(r) is positive then force is repulsive and if F(r) is negative then the force is attractive. Exemple of the attractive force is the gravitational forces between two mass points and the electrostatic force between two unlike charges. An exemple of the repulsive force is the electrostatic force between the like charges as in the case of the scattering of the alpha particles.

03-04-06

Exemple of central force :

(a) Gravitational Force :

Force on The Second particle m_2 due to The 1st particle m_1, can be written as

$$\vec{F}_{12} = \frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|^2} \times \text{unit vector in The direction of } (\vec{r}_1 - \vec{r}_2)$$

$$\vec{F}_{12} = \frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|^2} \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \quad \text{If } |\vec{r}_1 - \vec{r}_2| = r$$

Then

$$F_{12} = \frac{Gm_1m_2}{r^2} \frac{(\vec{r}_1 - \vec{r}_2)}{r} = \frac{Gm_1m_2}{r^3}\vec{r}$$

The corresponding Potential Energy will be given as

$$V_{(r)} = \frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{Gm_1m_2}{r}$$

(b) **Coloumb Force :**

The Columbic force is generally given as

$$F_c = \pm \frac{q_1q_2}{4\pi\epsilon_0} \frac{1}{r^2} = \pm \frac{K}{r^2}$$

$$\text{where } K = \frac{q_1q_2}{4\pi\epsilon_0}$$

then The corresponding Potential will be

$$V = \pm \frac{K}{r}$$

(c) **Hook's law :-** In The Hook's law the direction of force is always towards the means position.

$$F = -Kx$$

The corresponding potential is

$$V = \frac{1}{2}Kx^2 = \frac{c}{x^2} \quad \text{where } c = \frac{K}{2}$$

03-04-06

(d) Any Type of Force having Potential as $\frac{1}{r^n}$

The any type of force which have the potential in the form of $\frac{1}{r^n}$

For example in simple harmonic motion The Potential is $\frac{1}{r^2}$

The Potential of dipole moment is $\frac{\vec{P}.\vec{r}}{r^3}$

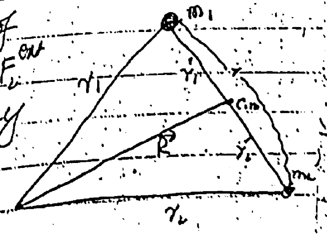
Here Potential does depends on The angle between $\vec{P}$ and $\vec{r}$

Generally we say That a force which always acts along a ~~fixed~~ point is called Central force.

* * *

REDUCTION OF TWO BODIES PROBLEM TO EQUIVALENT SINGLE BODY PROBLEM # 134

Consider the motion of two Particles of masses m_1 and m_2 separated by a distance r. Let F^{ext} be the total force which is acting on the system. Similarly let F^{int} be the total internal force due to the interaction between the two Particles. The total external force on The system will be the sum of external forces F_1^{ext} and F_2^{ext} which are acting separately on the two particles



Hence we have

$$F^{ext} = F_1^{ext} + F_2^{ext}$$

And The internal force which exist between these two Particles is given by Newton 3rd law which are action and reaction on each other

04-04-06

$\bar{F}_{12}^{int} = -\bar{F}_{21}^{int}$

individualy the total force on each body will be given as because it is two particle system.

$$F_{1total} = m_1\ddot{r}_1 = F_1^{ext} + F_{12}^{int} \quad — ①$$

and the force on the 2nd body will be

$$\bar{F}_{2Total} = m_2\ddot{r}_2 = F_2^{ext} + F_{21}^{int} \quad — ②$$

Adding eq ① and eq ② we have

$$m_1 r_1 + m_2 r_2 = F_1^{ext} + F_{12}^{int} + F_{12}^{ext} + F_{21}^{int}$$

$$(\text{As } F_{12}^{int} = -F_{21}^{int})$$

$$m_1\ddot{r}_1 + m_2\ddot{r}_2 = F_1^{ext} + F_2^{ext} + \cancel{F_{12}^{int}} - \cancel{F_{12}^{int}}$$

dividing & multiply the L.H.S by (m_1+m_2)

$$\frac{(m_1+m_2)(m_1\ddot{r}_1 + m_2\ddot{r}_2)}{(m_1+m_2)} = F_1^{ext} + F_2^{ext}$$

$$\Rightarrow M\ddot{R} = F^{ext} \quad — ③$$

where $F^{ext} = F_1^{ext} + F_2^{ext}$, $M = (m_1+m_2)$

and $\vec{R} = \frac{(m_1r_1+m_2r_2)}{(m_1+m_2)}$ ④ $\Rightarrow \ddot{R} = \frac{(m_1\ddot{r}_1+m_2\ddot{r}_2)}{(m_1+m_2)}$

Let the position vector of particle 1 relative to particle 2 is given as

$$r = r_1 - r_2 \quad — ⑤$$

solving eq ④ and eq ⑤ we have

As $r = r_1 - r_2 \Rightarrow r_2 = r_1 - r$

$\vec{R} = \frac{(m_1r_1+m_2r_2)}{(m_1+m_2)}$ putting the value of r_2 we have

04-04-06

$$R = \frac{m_1r_1 + m_2(r_1 - r)}{(m_1+m_2)} = \frac{m_1r_1 + m_2r_1 - m_2r}{(m_1+m_2)}$$

or $\vec{R} = \frac{(m_1+m_2)r_1}{(m_1+m_2)} - \frac{m_2r}{(m_1+m_2)}$

$$\Rightarrow R + \frac{m_2r}{(m_1+m_2)} - r_1 = 0$$

$$\Rightarrow \boxed{r_1 = R + \frac{m_2}{(m_1+m_2)}r}$$

Similarly $R = \frac{m_1r_1 + m_2r_2}{(m_1+m_2)}$ putting $r_1 = r + r_2$

$$R = \frac{m_1(r+r_2) + m_2r_2}{(m_1+m_2)} = \frac{m_1r + m_1r_2 + m_2r_2}{(m_1+m_2)}$$

$$R = \frac{(m_1+m_2)r_2}{(m_1+m_2)} + \frac{m_1r}{(m_1+m_2)}$$

$$\Rightarrow R = r_2 + \frac{m_1}{(m_1+m_2)}r$$

or $\boxed{r_2 = R - \frac{m_1}{(m_1+m_2)}r}$

In order to write eq ① and eq ② in term of $\bar{r}$ multiply eq ① by m_2 and multiply eq ② by m_1 and then Substract.

$$m_1m_2\ddot{r}_1 = m_2F_1^{ext} + m_2F_{12}^{int}$$

$$m_1m_2\ddot{r}_2 = m_1F_2^{ext} + m_1F_{21}^{int}$$

Substract the above from each other

$$m_1m_2(\ddot{r}_1 - \ddot{r}_2) = (m_2F_1^{ext} - m_1F_2^{ext}) + m_2F_{12}^{int} - m_1F_{21}^{int}$$

$$m_1m_2(\ddot{r}_1 - \ddot{r}_2) = (m_2F_1^{ext} - m_1F_2^{ext}) + m_2F_{12}^{int} + m_1F_{12}^{int}$$

$(\text{As } F_{12}^{int} = -F_{21}^{int}$

4-4-06

$$m_1m_2(\ddot{r}_1-\ddot{r}_2) = m_2F_1^{ext} - m_1F_2^{ext} + (m_1+m_2)F_{12}^{int}$$

$$m_1m_2(\ddot{r}_1-\ddot{r}_2) = (m_1+m_2)F_{12}^{int} + m_1m_2\left(\frac{F_1^{ext}}{m_1} - \frac{F_2^{ext}}{m_2}\right)$$

dividing both sides by (m_1+m_2)

$$\frac{m_1m_2}{(m_1+m_2)}(\ddot{r}_1-\ddot{r}_2) = F_{12}^{int} + \frac{m_1m_2}{(m_1+m_2)}\left(\frac{F_1^{ext}}{m_1} - \frac{F_2^{ext}}{m_2}\right)$$

$$\Rightarrow \boxed{\mu\ddot{r} = F_{12}^{int} + \mu\left(\frac{F_1^{ext}}{m_1} - \frac{F_2^{ext}}{m_2}\right)} \quad \text{—} ⑧$$

where $\mu = \frac{m_1m_2}{(m_1+m_2)}$ = which is called as reduced mass of Particles

and $\ddot{r} = \ddot{r}_1 - \ddot{r}_2$

The eq ⑧ will be $\mu\ddot{r} = F_{12}^{int}$ if

① If External force acting is zero means $F_1^{ext} = F_2^{ext} = 0$ OR

② If the external forces F_1^{ext} and F_2^{ext} are proportional to the masses of the particle on which they act and produce equal acceleration such that $\frac{F_1^{ext}}{m_1} = \frac{F_2^{ext}}{m_2}$

Under these two conditions eq ⑧ will be written as

$$\boxed{\mu\ddot{r} = F_{12}^{int}} \sim ⑨$$

which is the equation of motion of the particle having mass equal to reduced mass μ and moving under the action of force F_{12}^{int}. Such a reduction of two body system to one body system is very convenient and is used in classical as well as in quantum mechanics. The reduction is equivalent to replacing the system by a mass equal to reduced mass and considering the acceleration produced in it due to the interaction force.

4-4-06

Equation ⑨ and eq ③ represent the motion of the two body system under the action of the internal and external forces as long as the condition applied which are valid. The famous exemple of such force is gravitational force which is the exemple of 1st condition eq ⑨. The exemple of 2nd condition is the motion of moon around the Earth eq ⑨.

If the mass of one particle is extremely large as compared to the other $m_1 >> m_2$ then

$$\mu = \frac{m_1m_2}{m_1+m_2} = \frac{\cancel{m_1}m_2}{\cancel{m_1}(1+\frac{m_2}{m_1})}$$

$$\mu = \frac{m_2}{(1+\frac{m_2}{m_1})} = m_2 \qquad \frac{m_2}{m_1} \simeq 0$$

$$\Rightarrow \mu \simeq m_2$$

In this case the center of mass of the system concide with the center of mass of the heavier body of mass m_1. This is also used in Bohr theory of hydrogen atom or motion of settelite around the earth, or earth around the sun. This fact can be easily understood if we apply Newton second law of motion to these cases. Since mass m_2 is small force F_{21}^{int} produce appreciable acceleration in m_2 where as force F_{12}^{int} produce negligible acceleration in m_1. That is why " an apple appears to fall towards the earth and not the earth towards the apple ;

* * *

MOTION IN A CENTRAL FORCE FIELD OR THE EQUATION OF MOTION OF OR IST INTEGRAL SHOW THAT THE TOTAL ENERGY IN CENTRAL FORCE FIELD IS CONSTANT OF MOTION:

A force is said to be a central one if it acts along the position vector of a particle drawn from the center of force then it can be expressed as

$$F(r) = \hat{r} F(r)$$

where $\hat{r}$ is the unit vector along the direction of position vector $\vec{r}$ if $F(r)$ is +ve the force is repulsive if $F(r)$ is negative the force is attractive. Exemples are gravitational and electrostatic forces.

we now consider when only internal forces attractive or repulsive are acting on the particles.

Let the origin be the center of force and the particle of reduced mass is at a distance $\vec{r}$ from it which is the seperation distance between the two particles. The motion of this hypothetical particle in the central force, which is the force of interaction between the two particles, is equivalent to the two body motion.

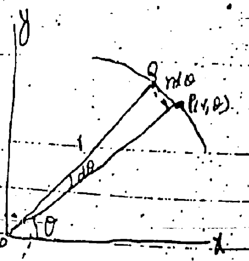


Figure 2

Potential energy V is the function of distance of this hypothetical particle

10-04-06

from the center of force only and does not depends upon the orientation Hence the system has a spherical symmetry. As a result of this the angular momentum of the particle is conserved. This is easily seen since the torque is

$$N = \frac{dL}{dt} = r \times F(r)$$
$$= r \times \hat{r} F(r)$$

As $r \times \hat{r} = 0$ Then

$$\frac{dL}{dt} = r \times \hat{r} F(r) = 0$$

$$\Rightarrow \quad dL = 0 \times dt = 0$$
$$\int dL = \int 0$$
$$\Rightarrow \quad L = \text{Const.}$$
$$\Rightarrow r \times P = \text{Const} \quad \text{—— Ⓐ}$$

From eq Ⓐ it is obvious that the plane containing r and P are alway perpendicular to L. Hence the component of L along any axis through the center of the force is constant. Further since there is no component of F(r) perpendicular to plane containing r and P, then the motion will always remain in the plane of r and P which is also the plane containing initial position and momentum.

We come accros a special case when L=0 - We come accros a special case when L=0 - and In this case r is perallel to P and the motion is along the straigth line.

Let us choose the x, y axies in the plane of motion with "O" is the origm let r and θ be the plane polar coordinates of a point where a particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is situated

0-04-06

Now the radial and Transverse acceleration are given by

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

"As we are going to find the Energy" the equation of motion of two body system when one of the body is at origin along 'r' and 'θ' will be

$$\left.\begin{array}{l} F(r) = \mu(\ddot{r} - r\dot{\theta}^2) \\ f(\theta) = \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{array}\right\} — ①$$

From the second equation

$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$
$$\mu r\ddot{\theta} + 2\mu\dot{r}\dot{\theta} = 0$$

multiplying by r we get

$$\mu r^2\ddot{\theta} + 2\mu r\dot{r}\dot{\theta} = 0$$
$$\frac{d}{dt}(\mu r^2\dot{\theta}) = 0$$
$$d(\mu r^2\dot{\theta}) = 0 \times dt = 0$$

Integrating we get

$$\int d(\mu r^2\dot{\theta}) = \int 0$$
$$\Rightarrow \quad \mu r^2\dot{\theta} = \text{Constt}$$

But $\mu r^2\dot{\theta} = L = \text{constant}$ — ②

Equation ② can be interpreted as the Kepler's second law which states that in any central force field the areal velocity is constant.

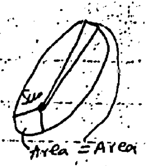
The radius vector drawn from the center of the force to the other body sweeps equal area in equal interval of time i-e the area swept out by a radius in time dt is

$$dA = \frac{1}{2}r \cdot r\,d\theta$$

10-04-06

$$d_A = \frac{1}{2}r^2 d\theta$$

Hence the rate at which the area is swept out or the areal velocity is given by



$$\dot{A} = \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$
$$\dot{A} = \frac{1}{2}r^2\dot{\theta} \quad — ③ \qquad \left(\text{As } \frac{d\theta}{dt} = \dot{\theta}\right)$$

Where $\dot{A}$ is the Areal velocity

As from eq ② $\mu r^2\dot{\theta} = L \Rightarrow \dot{\theta} = \frac{L}{\mu r^2}$

putting in eq ③ we have

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}r^2\left(\frac{L}{\mu r^2}\right) = \frac{L}{2\mu}$$
$$\dot{A} = \frac{L}{2\mu} = \text{Constant} \quad — ④$$

This gives $\dot{A}$ is constant thus the areal velocity in the central force field is constant which is Kepler's second law of planetry motion.

If the force is conservative then the total energy E of the particle is

$$E = T + V \qquad T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \qquad V = V(r)$$
$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r)$$

putting values of $\dot{\theta} = \frac{L}{\mu r^2}$ in above

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{L^2}{\mu^2 r^4}\right) + V(r)$$
$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \quad — ⑤$$

The fact that total Energy E for a central force is constant, can be derived from eq ①

$F_{(r)} = \mu(\ddot{r} - r\dot{\theta}^2)$

$F_{(r)} = \mu\ddot{r} - \mu r\dot{\theta}^2$

For conservative force $F_{(r)} = -\nabla V_{(r)}$

then $F_{(r)} = -\nabla V_{(r)} = \mu\ddot{r} - \mu r\dot{\theta}^2$

$-\frac{d}{dr}V_{(r)} = \mu\ddot{r} - \mu r\dot{\theta}^2$

As $\nabla = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

but here because V is only function of r

$\mu\ddot{r} = -\frac{d}{dr}V_{(r)} + \mu r\dot{\theta}^2$

putting $\dot{\theta} = \frac{L}{\mu r^2}$ in above we get

$\mu\ddot{r} = -\frac{d}{dr}V_{(r)} + \mu r\left(\frac{L}{\mu r^2}\right)^2$

$= -\frac{d}{dr}V_{(r)} + \cancel{\mu r}\frac{L^2}{\mu^2 r^4}$

$\mu\ddot{r} = -\frac{d}{dr}V_{(r)} + \frac{L^2}{\mu r^3}$

putting value of $\frac{1}{r^3}$

As $\frac{1}{r^3} = -\frac{d}{dr}\left(\frac{1}{2r^2}\right)$ then

$\mu\ddot{r} = -\frac{d}{dr}V_{(r)} + \frac{L^2}{\mu}\left(-\frac{d}{dr}\frac{1}{2r^2}\right)$

$= -\frac{d}{dr}V_{(r)} - \frac{d}{dr}\frac{L^2}{2\mu r^2}$

$\mu\ddot{r} = -\frac{d}{dr}\left[V_{(r)} + \frac{L^2}{2\mu r^2}\right]$ —⑥

multiplying both sides by $\dot{r}$ we get

$\mu\dot{r}\ddot{r} = -\dot{r}\frac{d}{dr}\left[V_{(r)} + \frac{L^2}{2\mu r^2}\right]$

As $\dot{r}\frac{d}{dr} = \frac{dr}{dt}\frac{d}{dr} = \frac{d}{dt} \Rightarrow \dot{r}\frac{d}{dr} = \frac{d}{dt}$

putting in above

$\mu\dot{r}\ddot{r} = -\frac{d}{dt}\left[V_{(r)} + \frac{L^2}{2\mu r^2}\right]$

As $\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = \mu\dot{r}\ddot{r}$ then above will be putting value of $\mu\dot{r}\ddot{r}$

$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}\left[V_{(r)} + \frac{L^2}{2\mu r^2}\right]$

$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) + \frac{d}{dt}\left[V_{(r)} + \frac{L^2}{2\mu r^2}\right] = 0$

$\frac{d}{dt}\left[\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r} + V_{(r)}\right] = 0.$

$d\left[\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r} + V_{(r)}\right] = 0 \times dt = 0$

Integrating we have

$\int \cancel{dt}\left[\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r} + V_{(r)}\right] = \int 0$

$\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r} + V_{(r)} = \text{Const}$

and it is equal to the total energy E

Thus we have

$\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r} + V_{(r)} = \text{Const} = E$

Thus the total Energy is constant of motion

The above equation can be written as

$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r} + V_{(r)}$

$\frac{1}{2}\mu\dot{r}^2 = E - V_{(r)} - \frac{L^2}{2\mu r}$

$\dot{r}^2 = \frac{2}{\mu}\left(E - V_{(r)} - \frac{L^2}{2\mu r}\right)$

Taking under root on both sides

$\Rightarrow \dot{r} = \left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r}\right)\right]^{1/2}$ —⑦

As $\dot{r} = \frac{dr}{dt}$ then

$\dot{r} = \frac{dr}{dt} = \left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r}\right)\right]^{1/2}$

or $dt = \frac{dr}{\left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r}\right)\right]^{1/2}}$ —⑧

12-04-06

The interval conditions are

When $t=0$ $r=r_0$

When $t=t$ $r=r$

integrating eq ⑧ we have

$$\int_0^t dt = \int_{r_0}^{r} \frac{dr}{\left[\frac{2}{\mu}\left(E-V-\frac{L^2}{2\mu r^2}\right)\right]^{1/2}}$$

$$t = \int_{r_0}^{r} \frac{dr}{\left[\frac{2}{\mu}\left(E-V-\frac{L^2}{2\mu r^2}\right)\right]^{1/2}} \quad — ⑨$$

The other equation for $\dot{\theta}$ will comes from

$$L = \mu r^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{L}{\mu r^2}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \frac{L}{\mu r^2}$$

$$d\theta = \frac{L}{\mu r^2}dt$$

Integrating above by condition

When $t=0 \Rightarrow, \theta=\theta_0$

When $t=t \Rightarrow \theta=\theta$

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{L}{\mu r^2}dt$$

or $$\theta-\theta_0 = \int_0^t \frac{L}{\mu r^2}dt$$

or $$\theta = \theta_0 + \int_0^t \frac{L}{\mu r^2}dt \quad — ⑩$$

The solutions of equation ⑨ & ⑩ are given in term of four constants viz (namely) E, L, θ_0 and r_0, when the initial velocity and the position are known.

12-04-06

The above method of obtaining the solution of equations of motions expressed both as "r" and "θ" are function of time we are very oftenly interested in the shape of the orbit i-e the way in which "r" depends upon θ.

To find this dependence we eliminate time from equation as

In order to eliminate time we consider

$$d\theta = \frac{d\theta}{dt}\cdot\frac{dt}{dr}\cdot dr = \frac{\dot{\theta}}{\dot{r}}dr.$$

putting value of $\dot{\theta}$ we get — As $L=\mu r^2\dot{\theta}$, $\dot{\theta} = \frac{L}{\mu r^2}$

$$d\theta = \frac{L/\mu r^2}{\dot{r}}dr = \frac{L/r^2}{\mu\dot{r}}dr$$

putting value of $\dot{r}$ from eq ⑦ we get

$$d\theta = \frac{L/r^2}{\mu\left[\frac{2}{\mu}\left(E-V-\frac{L^2}{2\mu r^2}\right)\right]^{1/2}}dr = \frac{L/r^2\,dr}{\left[2\mu\left(E-V-\frac{L^2}{2\mu r^2}\right)\right]^{1/2}}$$

Integrating we get

$$\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{L/r^2}{\left[2\mu\left(E-V-\frac{L^2}{2\mu r^2}\right)\right]^{1/2}}dr$$

or $$\boxed{\theta-\theta_0 = \int_{r_0}^{r} \frac{L/r^2}{\left[2\mu\left(E-V-L^2/2\mu r^2\right)\right]^{1/2}}dr} \quad — ⑪$$

We have obtained the formal solution of the form in which an integral can be solved easily in some specific forms of the law of force, the most important law of force is the inverse square law.

12-04-06

GENRAL FEATURES OF THE MOTION:-

MOTION "OF" IN Arbitrary POTENTIAL FIELD:-

"The equivalent one dimensional problem is discussed inside here"

Consider one dimensional motion in "r" with velocity we know that

$$\dot{r} = \left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)\right]^{\frac{1}{2}} \sim Ⓐ$$

As we know that radial component can be written as $F_{(r)} = \mu(\ddot{r} - r\dot{\theta}^2)$

$$F_{(r)} = \mu\ddot{r} - \mu r\dot{\theta}^2$$

or $\mu\ddot{r} = F_{(r)} + \mu r\dot{\theta}^2$

putting $\dot{\theta} = L/\mu r^2$ in above

$$\mu\ddot{r} = F_{(r)} + \mu r \times \frac{L^2}{\mu^2 r^4}$$

or $\mu\ddot{r} = F_{(r)} + \dfrac{L^2}{\mu r^3} \sim ①$

The $\dfrac{L^2}{\mu r^3}$ is traditionaly known as centrifugal force. It is not strictly a force since it does not ariges out of interaction between two particles. It arise due to accelerated motion of particle in the orbit is pseudo or false force.

The above equation ① can also be written as

$$-\nabla V_e = F_{(r)} + \frac{L^2}{\mu r^3}$$

where V_e is the effective Potential

or $-\nabla V_e = -\nabla V_{(r)} + \dfrac{L^2}{\mu r^3}$

we prove that the $\dfrac{L^2}{\mu r^3}$ is centrifugal force

As $L = \mu r^2\dot{\theta} = \mu r^2\omega$

As $\dot{\theta} = \omega$ then

$$\frac{L^2}{\mu r^3} = \frac{\mu^2 r^4\omega^2}{\mu r^3}$$

$$\frac{L^2}{\mu r^3} = \mu r\omega^2 = \mu r\left(\frac{v^2}{r^2}\right)$$

$$\frac{L^2}{\mu r^3} = \frac{\mu v^2}{r}$$

It direction is +ve ∴ it is centrifugal

12-04-06

where $F_{(r)} = -\nabla V_{(r)} = -\dfrac{d}{dr}V_{(r)}$

then

$$-\nabla V_e = -\nabla V_{(r)} + \frac{L^2}{\mu r^3}$$

or $\nabla V_e = \nabla V_{(r)} - \dfrac{L^2}{\mu r^3}$

or $\dfrac{d}{dr}V_e = \dfrac{d}{dr}V_{(r)} - \dfrac{L^2}{\mu r^3}$

multiplying by dr we get

$$\frac{d}{dr}V_e(dr) = \frac{d}{dr}V_{(r)}(dr) - \frac{L^2}{\mu r^3}dr$$

integrating both sides we get

$$-\int d(V_e) = \int d\,V_{(r)} - \int \frac{L^2}{\mu r^3}dr$$

or $V_e = V_r - \dfrac{L^2}{\mu}\displaystyle\int r^{-3}dr$

$$\boxed{V_e = V_r + \frac{L^2}{2\mu r^2}} — ②$$

$\dfrac{L^2}{2\mu r^2}$ is the Potential Energy arising as a result of centrifugal force

Further more eq ②. can be written as

$$V_e = V + \frac{L^2}{2\mu r^2}$$

As we know that from previous topic

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V$$

at $V_{(\infty)} = 0$ then putting value of V_e

or $E = \frac{1}{2}\mu\dot{r}^2 + V_e$

Now the equation Ⓐ can be written as

$$\dot{r} = \left[\frac{2}{\mu}\left(E - \left(V + \frac{L^2}{2\mu r^2}\right)\right)\right]^{\frac{1}{2}}$$

then

$$\dot{r} = \left[\frac{2}{\mu}(E - V_e)\right]^{\frac{1}{2}} — Ⓑ$$

17-04-08

Squaring both sides we have

$$\dot{r}^2 = \frac{2}{\mu}(E - V_e)$$

$$\text{or} \quad \frac{1}{2}\mu \dot{r}^2 = E - V_e \quad —③$$

Let us suppose a particle which has total energy E. The arbitrary potential field is represented by the curve drawn in Contmas line. The total energy E is equal to the V_e at the points r_1, r_2 & r_3

As at this points $E = V_e$ or $E - V_e = 0$ putting in eq ③ $\Rightarrow v_r = \dot{r} = 0$ such points as the radial velocity is zero here, such points are called turning points.

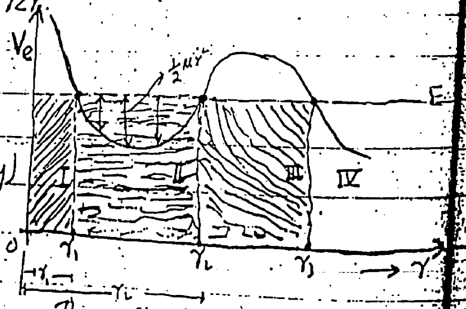
At all other points there exist some difference between E and V_e

The whole range of the value of r can be divided into various regions we will discuss one by one.

Ⓘ Region (I) For which $r < r_1$

In this region the Potential Energy V_e is greater than E

Hence $\frac{1}{2}\mu \dot{r}^2$ (Kinetic energy) will be negative and velocity $\dot{r}$ (v) will be imaginary. The region is forbidden for the particle, the particle would never be in such region.



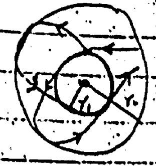
Ⓘⓘ Region (II) For which $r_1 < r < r_2$:-

In this region the total energy E is greater than the potential Energy V_e

At r_1 & r_2 $E = V_e$ therefore these

17-04-06.

are called turning points. The distances r_1 and r_2 are called the apsidal distance. The motion of the particle is oscillatory in the potential well and the particle will confine to this region. The particle does not possess enough kinetic energy to cross the potential (well) barriers



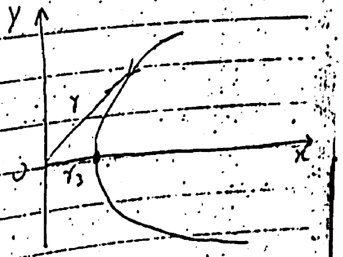
Ⓘⓘⓘ Region (III) For which $r_2 < r < r_3$

He the Total energy E is also smaller then the potential Energy V_e

Hence this region is forbidden for the particle.

(IV) Region (IV) :- For which $r \geq r_3$

This region has only one turning point which is at $r = r_3$. A particle initially at infinite distance from the center of force is able to approach upto $r = r_3$ where it gets rebounded and proceeds (unhindered) unhindered to infinity again such a phenomenenon is called Scattering.



In the region (II) the motion is bounded and in the region (IV) it is unbounded.

∴ If the total energy of the particle in the conservative field is zero

$$E = T + V_e = 0$$

$$T = -V_e$$

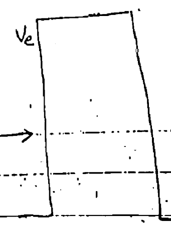
then the particle situated at any point will

17-04-06

move towards infinity. The velocity obtained by the particle due to This relation is called escape velocity.

* *

General about The Former Topic

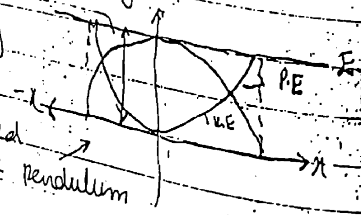


Consider a particle which have energy E and it is approaching towards the barrier which have energy Ve

Here $E < V_e$, Then classicaly the Particle has no possibility that it would be pass through the barrier. The existance of the particle in the other side of the barrier is zero classically But quantum mechanically the particle has the possibility that it would be pass through it. The particle has alway the Prababolity of penetrating the potential barrier. This is so called the Tunneling effect. wave is attached to any body due to De Broglie hypothesis by $\lambda = \frac{h}{mv} = \frac{h}{p}$

* Draw the graph of potential Energy, kinetic energy and the sum of kinetic and potential which is total Energy.

Draw the graph of Simple harmonic motion of Mass attached spring or the simple pendulum



MOTION IN AN INVERSE SQUARE LAW FORCE FIELD :-

19.04.06

The inverse square law of force is that force which varies as the square of the distance between two Particles (Bodies)

i.e $F_{(r)} = \frac{K}{r^2}$

where K is a constant of proportionality which depends upon the nature of physical problem

As we know that $F = -\frac{dv}{dr}$

putt in above

$$-\frac{dv}{dr} = \frac{K}{r^2}$$ where V is a scalar function

$$-dV = \frac{K}{r^2} dr$$

Integrating we have

$$-\int dv = K \int r^{-2} dr = K \frac{r^{-2+1}}{-2+1} = -\frac{K}{r}$$

$$\cancel{-} V_{(r)} = \cancel{-}\frac{K}{r}$$

$$\Rightarrow V_{(r)} = \frac{K}{r}$$

Then the effective potential in the inverse square field is $V_e = V_{(r)} + \frac{L^2}{2\mu r^2}$

putting value of $V_{(r)}$ we get

$$V_e = \frac{K}{r} + \frac{L^2}{2\mu r^2} \quad \text{Ⓐ}$$

Two important inverse square law are the gravitational and coulomb law

In gravitational law $K = -Gm_1m_2$

Where the -ve sign represent that the two bodies is attractive force between the two m₁ + m₂ are masses of the two bodies

19-04-06

In case of Coulomb law $K = K_0 = \pm \frac{q_1 q_2}{4\pi\epsilon_0}$

Here K is -ve if force between two charge is attractive and +ve if the force is repulsive.

The graph between effective potential Ve and r can be plotted for varies values of K.

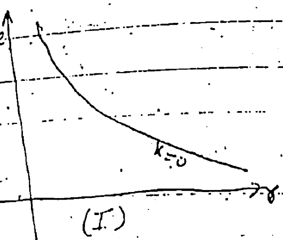
Case (I) when $K=0$

then eq (A) ⇒

$$V_e = \frac{L^2}{2\mu r^2}$$

$$V_e \propto \frac{1}{r^2}$$

The graph will be

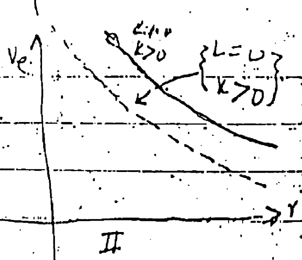


case (II) when $K>0$ and $L=0$ then eq (1)

$$V_e = \frac{K}{r} + \frac{0}{2\mu r^2}$$

$$V_e = \frac{K}{r}$$

$$\Rightarrow V_e \propto \frac{1}{r}$$

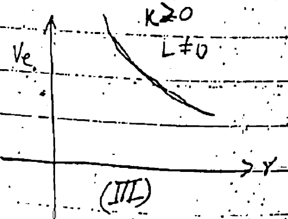


Case (III) when $K>0$ & $L\neq 0$

then

$$V_e = \frac{K}{r} + \frac{L^2}{2\mu r^2}$$

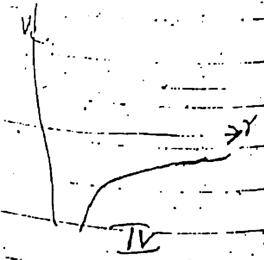
then graph will be



Case (IV) when $K<0$ = -ve then

$$V_e = -\frac{K}{r} + \frac{L^2}{2\mu r^2}$$

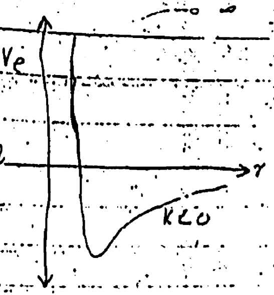
The graph will be



19-04-06

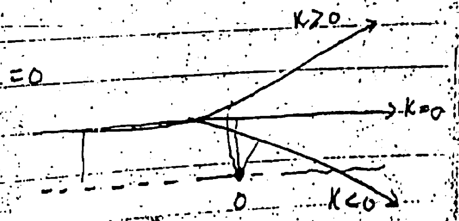
Then Combining the case I & IV we have the graph will be

For repulsive force $K \geq 0$ the P.E (Ve) curve does not have a minimum as for the similar curve for a force of attraction $K<0$.



★ A particle having any arbitrary value of energy approaches the center of force from infinity ($r=\infty$), reaches the closest distance of approach turns around and again moves to infinity. The nature of the orbit is shown for the positive and negative K. This orbits have unbounded motion.

★ The orbit for $K=0$ is just a straight line. There is no force existing here.



★ If the force is attractive the effective potential is

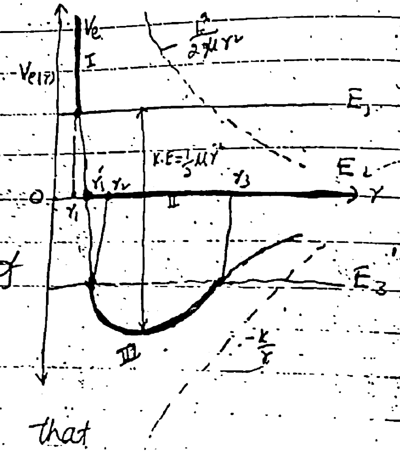
$$V_e = -\frac{|K|}{r} + \frac{L^2}{2\mu r^2}$$

In this case the orbit is around the center of force "0".

★ If the force is repulsive the particle would follow an orbit along which it is deflected away from the center of force "0".

24-04-06

If we plot the two terms of potential energy viz $-\frac{|K|}{r}$ and $\frac{L^2}{2\mu r^2}$ against the distance r.
The variation of resultant V_e with respect to r is shown by the continous curve. The range in which r can very is $0 \le r < \infty$.
Now we have special cases about the curve



Special cases:-

Case 1:-
For total Energy E_1 we get intersection of the straight line $E_1 = \text{constt}$ with P.E curve at r_1. This corresponds to the case that radial component of K.E is zero. r_1 is the real root of the Equation

$$E_1 = V_e = -\frac{|K|}{r} + \frac{L^2}{2\mu r^2}$$

This only one turning point at $r = r_1$ which corresponds to the scattering unbounded motion.

Case 2:- If the Energy of the particle has value $E_2 = 0$ we have a similar case, the two roots of the equation $r = r_1'$ and $r = \infty$, the particle goes to infinity but its radial velocity falls off continously and become zero at infinity. This also shows the scattering process.

26-04-08

putting $F(\frac{1}{u}) = |K| u^2$ in eq ③ we have

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu r^2}{L^2} \times |K| u^2$$

Then putting in above As $r^2 = \frac{1}{u^2}$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{u^2 L^2}|K| u^2 = \frac{\mu |K|}{L^2}$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{|K|\mu}{L^2} \quad \text{④}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + \left(u - \frac{|K|\mu}{L^2}\right) = 0 \quad \text{(4A)}$$

Let putt we have $y = \left(u - \frac{|K|\mu}{L^2}\right)$

Then $\frac{d^2y}{d\theta^2} = \frac{d^2u}{d\theta^2} - \cancel{\left(\frac{d^2}{d\theta^2}\overset{\text{constant}}{\frac{|K|\mu}{L^2}}\right)}$

$$\frac{d^2y}{d\theta^2} = \frac{d^2u}{d\theta^2}$$

putting these values in eq (4A) we have

$$\frac{d^2y}{d\theta^2} + y = 0 \quad \text{⑤}$$

This is a second order differential equation in y as a function of θ and has a solution

$$y = A\cos(\theta - \theta_0) \quad \text{(5B)}$$

Where A and θ_0 are constants of integration.

As we know that $y = u - \frac{|K|\mu}{L^2}$ — (5A)

Comparing eq (5B) and (5A) we have

$$u - \frac{|K|\mu}{L^2} = A\cos(\theta - \theta_0)$$

26-04-06

$$u = \frac{|K|\mu}{L^2} + A\cos(\theta-\theta_0) \qquad \text{As } u = \frac{1}{r}$$

$$\frac{1}{r} = \frac{|K|\mu}{L^2} + A\cos(\theta-\theta_0)$$

$$\frac{1}{r} = \frac{|K|\mu}{L^2}\left[1 + \frac{L^2A}{|K|\mu}\cos(\theta-\theta_0)\right]$$

$$\frac{L^2/\mu|K|}{r} = \left[1 + \frac{L^2A}{|K|\mu}\cos(\theta-\theta_0)\right] \quad — ⑥$$

Equation no.⑥ is similar to the eq

$$\frac{l}{r} = 1 + \epsilon\cos\theta \quad — ⑦$$

Comparing both ⑥ & ⑦ we have

Equation ⑦ is the equation of conic section with origin at the focus

$$\Rightarrow \quad l = \frac{L^2}{\mu|K|} = \text{semi latus rectum}$$

$$\Rightarrow \quad \epsilon = \frac{L^2A}{\mu|K|} = \text{eccentricity}$$

The same result can also be obtained by integrating.

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{(L/r^2)\,dr}{\left[2\mu\left(E - V - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}} \quad — Ⓐ$$

For the attractive inverse square law force field we have

$$V = \frac{-|K|}{r}$$

Then putting this value in eq Ⓐ we have

02-05-06

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{(L/r^2)\,dr}{\left[2\mu\left(E + \frac{|K|}{r} - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}}$$

Let $u = \frac{1}{r} \Rightarrow du = -\frac{1}{r^2}dr$

$$\Rightarrow dr = -r^2 du$$

$$\Rightarrow dr = -\frac{1}{u^2}du$$

Then above eq will be

$$\theta - \theta_0 = \int \frac{L\cancel{u^2}\left(-\frac{1}{\cancel{u^2}}du\right)}{\left[2\mu\left(E + Ku - \frac{u^2L^2}{2\mu}\right)\right]^{1/2}}$$

$$= -\int \frac{du}{\left[\frac{2\mu}{L^2}\left(E + Ku - \frac{u^2L^2}{2\mu}\right)\right]^{1/2}}$$

$$= -\int \frac{du}{\left[\frac{2\mu E}{L^2} + \left(\frac{2\mu K}{L^2}\right)u - \frac{2\mu}{L^2}\times\frac{u^2L^2}{2\mu}\right]^{1/2}}$$

$$\theta - \theta_0 = -\int \frac{du}{\left[\frac{2\mu E}{L^2} + \left(\frac{2\mu K}{L^2}\right)u - u^2\right]^{1/2}} \quad Ⓑ$$

As we know that solution of such type of integral is by using standard result,

$$\int \frac{dx}{(a + bx + cx^2)^{1/2}} = \frac{1}{\sqrt{-c}}\cos^{-1}\left(-\frac{b + 2cx}{\sqrt{b^2 - 4ac}}\right)$$

If we use this method on eq Ⓑ then the

$$c = -1, \quad b = \frac{2\mu K}{L^2}, \quad a = \frac{2\mu E}{L^2}$$

and

$$\theta - \theta_0 = -\int \frac{du}{(a + bu + u^2)^{1/2}}$$

02-05-06

$$\theta-\theta_0=-\left[\frac{1}{\sqrt{-(-1)}}\cos^{-1}\left(-\frac{\frac{2\mu k}{L^2}+2(-1)u}{\sqrt{\frac{4\mu^2k^2}{L^4}+\frac{8\mu E}{L^2}}}\right)\right]$$

$$\theta-\theta_0=-\left[\cos^{-1}\left(\frac{-\left(\frac{2\mu k}{L^2}-2u\right)}{\sqrt{\frac{4\mu^2k^2}{L^4}+\frac{8\mu E}{L^2}}}\right)\right]$$

$$\theta_0-\theta=\cos^{-1}\left(\frac{-\left(\frac{2\mu k}{L^2}-2u\right)}{\frac{2\mu k}{L^2}\sqrt{1+\frac{2EL^2}{\mu k^2}}}\right)$$

$$\cos(\theta_0-\theta)=\cos(\theta-\theta_0)=\frac{-\frac{2\mu k}{L^2}+2u}{\frac{2\mu k}{L^2}\sqrt{1+\frac{2EL^2}{\mu k^2}}}$$

$$\frac{2\mu k}{L^2}\left(1+\frac{2EL^2}{\mu k^2}\right)^{1/2}\cos(\theta-\theta_0)=-\frac{2\mu k}{L^2}+2u$$

$$\frac{2\mu k}{L^2}\left(1+\frac{2EL^2}{\mu k^2}\right)^{1/2}\cos(\theta-\theta_0)+\frac{2\mu k}{L^2}=2u$$

$$\frac{\cancel{2}\mu k}{L^2}\left[1+\left(1+\frac{2EL^2}{\mu k^2}\right)^{1/2}\cos(\theta-\theta_0)\right]=\cancel{2}u$$

$$\Rightarrow u=\frac{1}{r}=\left[1+\left(1+\frac{2EL^2}{\mu k^2}\right)^{1/2}\cos(\theta-\theta_0)\right]\frac{\mu k}{L^2}\quad — ⓒ$$

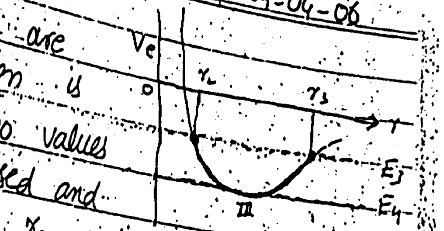
$$\Rightarrow \frac{1}{r}=\left[\frac{\mu|k|}{L^2}\pm\frac{\mu|k|}{L^2}\left(1+\frac{2EL^2}{\mu k^2}\right)^{1/2}\cos(\theta-\theta_0)\right]\quad — ⓓ$$

Since the cosine of an angle varies between the maximum and minimum which are 1 and -1, than the turning points will be given as

24-04-06

Case 3 :-

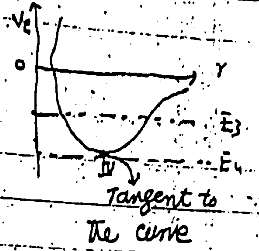
For $E_3 < 0$ the roots are r_2 and r_3. The motion is bounded between two values The orbits are closed and elliptic. The distance r_2 and r_3 gives positions of pericenter and apocenter of the orbit respectively.



Case 4 :-

For total Energy E_4, the two roots of the equations are

$$V_e=-\frac{|k|}{r}+\frac{L^2}{2\mu r^2}$$



Coincide and resulting orbit is circular ∴ The straight line E_4 = constt is tangential to the potential curve then

$$\frac{dV_e}{dr}=0 \text{ at the point of contact}$$

Then as we know that

$$V_e=V_{(r)}+\frac{L^2}{2\mu r^2}$$

where $V_{(r)}$ is any potential in the form $V=V(r)$

Then

$$\frac{dV_e}{dr}=\frac{d}{dr}V_{(r)}+\frac{d}{dr}\frac{L^2}{2\mu r^2}$$

As $\frac{dV_e}{dr}=0$ then

$$\frac{d}{dr}V_{(r)}+\frac{d}{dr}\frac{L^2}{2\mu r^2}=0$$

$$\frac{d}{dr}V-\frac{\cancel{2}L^2}{\cancel{2}\mu r^3}=0$$

$$r\quad -\frac{dV_{(r)}}{dr}=-\frac{L^2}{\mu r^3}$$

25.04.06

As we know that $F = -\frac{dV}{dr}$ then

$$F_{(r)} = -\frac{L^2}{\mu r^3}$$

The $F_{(r)}$ is just seen to be equal to the centripetal force require to obtain the circular motion, around the center of force.

The centripetal force main the circular orbit.

The r_2 and r_3 in case III are called Perigee and apogee in case of motion of earth respectively.

They are called perhelion and aphelion respectrie in case of motion of any planet around the sun.

$$F(r) = -\frac{L^2}{\mu r^3}$$

As $L = \mu r^2\dot{\theta}$ putt in above

$$F(r) = \frac{-\mu^2 r^4 \dot{\theta}^2}{\mu r^3}$$

$$F(r) = -\mu r \dot{\theta}^2$$

As $\dot{\theta} = \omega$ then

$$F(r) = -\mu r \omega^2$$

As $v = r\omega$

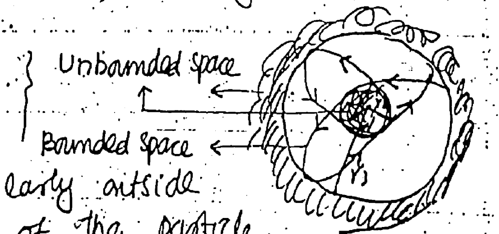
$\omega = \frac{v}{r}$ then

$$F(r) = -\mu r \frac{v^2}{r^2}$$

$$F(r) = -\frac{\mu v^2}{r}$$

which is centripetal Force

The insider r_1 the motion of the particle is unbounded. Similarly outside r_3 the motion of the particle is also unbounded. Only the particle can move in the space r_3 which is bounded and the motion is circular.



* *

MATI ULLAH 0345-9798233
M.Sc (previous)
Physics Department UOP
Village Jahangir Lalozai p/o Torkey
Surrani Bannu

EQUATION OF THE ORBIT :-

26-04-06

we know that equation of motion of a particle subjected to a central force is

$$\mu\ddot{r} = F(r) + \frac{L^2}{\mu r^3} \quad ①$$

we know putt this equation in a form which is suitable to carry out the study of the nature of the orbit. For this we consider

$$u = \frac{1}{r}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\cdot\frac{dt}{d\theta}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \qquad \text{where } \dot{\theta} = \frac{d\theta}{dt}, \quad \dot{r} = \frac{dr}{dt}$$

But we know that $\mu r^2\dot{\theta} = L$

$\dot{\theta} = \frac{L}{\mu r^2}$ putt in above

$$\frac{du}{d\theta} = -\frac{1}{r^2}\dot{r}\cdot\frac{\mu r^2}{L}$$

$$\frac{du}{d\theta} = -\frac{\mu\dot{r}}{L}$$

Further we have

$$\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-\frac{\mu\dot{r}}{L}\right)$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{dt}\left(-\frac{\mu\dot{r}}{L}\right)\frac{dt}{d\theta} \qquad \because \frac{dt}{d\theta} = \frac{1}{\dot{\theta}}$$

$$= \frac{d}{dt}\left(-\frac{\mu\dot{r}}{L\dot{\theta}}\right)$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu\ddot{r}}{L\dot{\theta}}$$

26-04-06

$$\frac{d^2u}{d\theta^2} = \frac{-\mu\ddot{r}}{\dot{\theta}L}$$

$$\Rightarrow -\mu\ddot{r} = \dot{\theta}L\frac{d^2u}{d\theta^2}$$

$$\text{or} \quad \mu\ddot{r} = -\dot{\theta}L\frac{d^2u}{d\theta^2} \quad — ②$$

Substituting the value of $\mu\ddot{r}$ in eq ① we have

$$-\dot{\theta}L\frac{d^2u}{d\theta^2} = F(r) + \frac{L^2}{\mu r^3}$$

As $r = \frac{1}{u}$

$$-\dot{\theta}L\frac{d^2u}{d\theta^2} = F(\tfrac{1}{u}) + \frac{L^2}{\mu r^3}$$

$$\frac{d^2u}{d\theta^2} = -\frac{F(\frac{1}{u})}{\dot{\theta}L} - \frac{L^2/\mu r^3}{\dot{\theta}L}$$

As $\dot{\theta} = \frac{L}{\mu r^2} \Rightarrow \dot{\theta}L = \frac{L^2}{\mu r^2}$,

putting in above we have.

$$\frac{d^2u}{d\theta^2} = \frac{-F(\frac{1}{u})}{L^2/\mu r^2} - \frac{L^2}{\mu r^3}\cdot\frac{\mu r^2}{L^2}$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu r^2}{L^2}F(\tfrac{1}{u}) - \frac{1}{r} \qquad \text{As } \frac{1}{r} = u$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu r^2}{L^2}F(\tfrac{1}{u}) - u$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu r^2}{L^2}F(\tfrac{1}{u}) \quad — ③$$

If the force field obeys the inverse square law then

$$F(r) = F(\tfrac{1}{u}) = \frac{|k|}{r^2}$$

$$\text{or} \quad F(\tfrac{1}{u}) = |k|u^2 \qquad \text{as } u = \frac{1}{r}$$

02-05-06

$$\frac{1}{r_1} = \frac{\mu|k|}{L^2} + A \quad \text{and} \quad \frac{1}{r_2} = \frac{\mu|k|}{L^2} - A$$

where the value of A is

$$A = \frac{\mu|k|}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}} \quad — ⓓ$$

Then the eccentricity will be given as

$$\epsilon = \frac{L^2 A}{\mu|k|}$$

By putting the value of A we have

$$\epsilon = \frac{L^2}{\mu|k|} \times \frac{\mu|k|}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

$$\Rightarrow \epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}} \quad — ⓔ$$

The equation ⓒ can be written as

$$\frac{1}{r} = \left[1 + \sqrt{\left(1 + \frac{2EL^2}{\mu k^2}\right)}\cos(\theta-\theta_0)\right]\frac{\mu k}{L^2}$$

$$\frac{1}{r} = \left[1 + \epsilon\cos(\theta-\theta_0)\right]\frac{\mu k}{L^2}$$

$$\Rightarrow \frac{(L^2/\mu k)}{r} = \left[1 + \epsilon\cos(\theta-\theta_0)\right] — ⓖ$$

As we know that

$$A = \frac{\mu|k|}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

$$\& \quad \epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

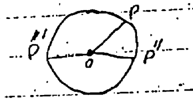
The nature of the orbit depends upon ϵ values of E (total Energy) we have a Table on the other page

03-05-06

SNO.	value of E	eccentricity(ϵ)	Nature of orbit
①	$E>0$	$\epsilon>1$	Hyberbolic
②	$E=0$	$\epsilon=1$	Parabolic
③	$V_e<E<0$	$0<\epsilon<1$	elliptical
④	$E=V_e$	$\epsilon=0$	circular

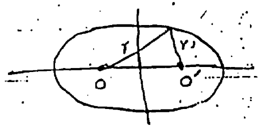
Circle:- The set of Points which are at equidistance from a fixed point called the origin of the circle

$OP = OP' = OP'' =$ constt



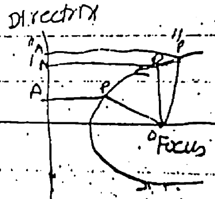
Ellipse:-
The distance of any point or the sum of the distance to any point from the two foci remain constt.

$r + r' =$ constt

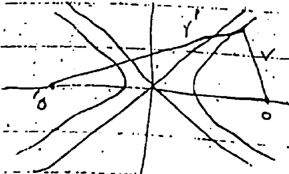


Parabola:-
The distance between the point on the Parabola and the Focus and similarly distance between directrix and points on parabola should be equal

$AP = OP$
$AP' = OP'$



Hyberbola:-
The distance between the points on hyberbola and the two foci remain constants



Elliptic Orbits:

08-05-06

In order to write the equation of the orbit in the polar form we write

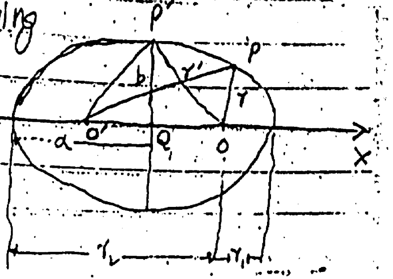
$$\frac{1}{r} = c(1+\epsilon\cos\theta) \quad ①$$

In eq ① we have taken $\theta_0 = 0$ and $c = \frac{\mu|k|}{L^2}$

The ellipse is a curve traced out by a particle moving in such a way that the sum of its distances from the two fixed foci O and O' is always constant

$OP + O'P =$ constt

$r + r' = 2a$ const (definition of ellipse)



where a is the semi major axis.

From the figure the distance between the turning points is given by

$$r_1 + r_2 = 2a \quad ②$$

As we know from the equation ①

$$\frac{1}{r_1} = c(1+\epsilon\cos\theta)$$

$$\Rightarrow r_1 = \frac{1}{c(1+\epsilon\cos\theta)} \quad ⓐ$$

&

$$\frac{1}{r_2} = c(1-\epsilon\cos\theta)$$

$$\Rightarrow r_2 = \frac{1}{c(1-\epsilon\cos\theta)} \quad ⓑ$$

The eq ⓐ & ⓑ are for general case. For the particular case they can be written as

$$r_1 = \frac{1}{c(1+\epsilon)} \quad \& \quad r_2 = \frac{1}{c(1-\epsilon)}$$

08-05-06

putting the values of r_1 and r_2 in equation ② we have

$$r_1+r_2 = \frac{1}{c(1+\epsilon)} + \frac{1}{c(1-\epsilon)} = 2a$$

or
$$\frac{(1-\cancel{\epsilon})+1+\cancel{\epsilon}}{c(1+\epsilon)(1-\epsilon)} = 2a$$

$$\frac{2}{c(1-\epsilon^2)} = 2a$$

$$\left.\begin{aligned} &\Rightarrow a = \frac{1}{c(1-\epsilon^2)} \\ &\text{or } c = \frac{1}{a(1-\epsilon^2)} \end{aligned}\right\} \text{—— ③}$$

putting the value of c in eq. ① we have

$$\frac{1}{r} = c(1+\epsilon\cos\theta)$$

$$\frac{1}{r} = \frac{1}{a(1-\epsilon^2)} \times (1+\epsilon\cos\theta)$$

$$\Rightarrow \frac{1}{r} = \frac{(1+\epsilon\cos\theta)}{a(1-\epsilon^2)}$$

$$\Rightarrow r = \frac{a(1-\epsilon^2)}{(1+\epsilon\cos\theta)} \text{ —— ④}$$

The equation ④ is the polar form of the orbit.

From the figure the distances between the foci is

$$OO' = r_2 - r_1$$

putting the value of r_1 & r_2 we have,

08-05-06

$$OO' = \frac{1}{c(1-\epsilon)} - \frac{1}{c(1+\epsilon)} = \frac{(1+\epsilon)-(1-\epsilon)}{c(1-\epsilon)(1+\epsilon)}$$

$$OO' = \frac{\cancel{1}+\epsilon-\cancel{1}+\epsilon}{c(1-\epsilon^2)} = \frac{2\epsilon}{c(1-\epsilon^2)}$$

or
$$OO' = \frac{2\epsilon}{c(1-\epsilon^2)} = 2a\epsilon \text{ (From The ellipse equation)}$$

If The Particle is situated at P' such that $OP' = O'P'$ it distance from O or O' must be equal to "a" since $OP' + O'P' = 2a$.

Then from the figure the semi minor axis 'b' is given by

$$(O'P')^2 = (O'Q)^2 + (QP')^2$$

As $O'Q = a\epsilon$, $O'P' = a$

and $P'Q = b$

Then putting these values in above equation we have

$$(O'P')^2 = (O'Q)^2 + (QP')^2$$

$$a^2 = (a\epsilon)^2 + b^2$$

or $b^2 = a^2 - a^2\epsilon^2$

$$b^2 = a^2(1-\epsilon^2)$$

or $b = a\sqrt{(1-\epsilon^2)}$ —— ⑤

As $OO' = 2a\epsilon$

$O'Q = a\epsilon$

As $OP' + O'P' = 2a$

when $OP' = O'P$

Then $2OP' + O'P' = 2a$

$2O'P' = 2a$

$O'P' = a$

And $P'Q = b$

The ellipse will be converted into circle when $b = a$

$\epsilon = 0 \Rightarrow$ Then As $\epsilon = \sqrt{1+\frac{2EL^2}{\mu K^2}} = 0$

when $\epsilon = 0$

$$\Rightarrow 1 + \frac{2EL^2}{\mu K^2} = 0$$

$$-\frac{2EL^2}{\mu K^2} = 1$$

09-05-06

$$\Rightarrow E = \frac{-\mu K^2}{2L^2} \quad \text{—⑥}$$

we know that $a = \frac{1}{c(1-\epsilon^2)}$ when $\epsilon = 0$ Then

As when $\epsilon = 0$ Then the radius of the resulting circle is given by

$$a = \frac{1}{c} \qquad \text{As } c = \frac{\mu |K|}{L^2}$$

Then $a = \frac{1}{c} = \frac{L^2}{\mu |K|} = \frac{|K|}{2E}$

KEPLER'S LAWS OF PLANETARY MOTION

Kepler's announced the three laws of planetry motion based on the observation before Newton laws of motion was formulated.

The keplar laws of Planetry motions are

1ST LAW:- The Planet moves around the Sun in a elliptical orbit with The Sun at one of The Foci of The orbit.

2ND LAW:- The area swept out by the radius vector drawn from the center of the Sun to the center of The planet in equal interval of time are equal. In other words the Areal velocity is constant.

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{Constt}$$

09-05-06

3rd LAW:- The square of the period of revolution of the planet is directly proportional to the cube of the semi-major axis of the orbit.

$$T^2 \propto a^3$$

$$\text{or } \frac{T^2}{a^3} = \text{Constt}$$

where T is the period of revolution

STATE AND PROVE KEPLER'S 3rd Law

STATEMENT:-

The square of the period of revolution of the planet is directly proportional to the cube of the semi major axis of the orbit. mathamatically

$$T^2 \propto a^3$$

$$\text{or } \frac{T^2}{a^3} = \text{Constt}$$

Proof:-

Generally we know that the period of area swept is

$$T = \frac{\text{area}}{\text{areal velocity}}$$

Area of circle $= \pi r^2$
Area of ellipse $= \pi ab$

We know That area of the ellipse is $= \pi ab$

And Areal velocity $= \frac{L}{2\mu}$

Then The Time period will be given by putting the values in above equation

$$T = \frac{\text{area}}{\text{Areal velocity}} = \frac{\pi ab}{L/2\mu}$$

$$T = \frac{2\pi \mu ab}{L} \quad \text{—①}$$

10-05-06.

But we know that for an ellipse

$$b = a\sqrt{1-\epsilon^2}$$

putting the value of b in eq ①

$$T = \frac{2\pi\mu ab}{L} = \frac{2\pi\mu a \cdot a\sqrt{1-\epsilon^2}}{L}$$

$$T = \frac{2\mu\pi a^2\sqrt{1-\epsilon^2}}{L} \quad \text{——} ②$$

But we have

$$a = \left|\frac{k}{2E}\right| = \frac{L^2}{\mu k}\frac{1}{(1-\epsilon^2)}$$

As, we know that

$$\epsilon = \sqrt{1+\frac{2EL^2}{\mu k^2}} \quad \text{——} ③$$

Squaring both sides we have

$$\epsilon^2 = 1 + \frac{2EL^2}{\mu k^2}$$

$$\text{or} \quad -\frac{2EL^2}{\mu k^2} = 1-\epsilon^2$$

$$\text{or} \quad \frac{L^2}{\mu k}\left|\frac{2E}{k}\right| = 1-\epsilon^2$$

As from above eq ③ $\left|\frac{2E}{k}\right| = \frac{1}{a}$ then putting the value above

$$\frac{L^2}{\mu k}\cdot\frac{1}{a} = (1-\epsilon^2)$$

$$\text{or} \quad \frac{L^2}{\mu k} = a(1-\epsilon^2) \quad \text{——} ④$$

Squaring eq ② we have

$$T^2 = \frac{4\mu^2\pi^2a^4(1-\epsilon^2)}{L^2}$$

10-05-06

$$\text{or} \quad T^2 = \frac{4\pi^2\mu^2 a^3}{L^2}\cdot a(1-\epsilon^2)$$

putting the value of $a(1-\epsilon^2)$ from eq ④ we have

$$T^2 = \frac{4\pi^2\mu^2a^3}{L^2} \times \frac{L^2}{\mu k}$$

$$T^2 = \frac{4\pi^2\mu}{k}a^3$$

$$\text{or} \quad T^2 = \text{Constt}\ a^3$$

$$\text{or} \quad \frac{T^2}{a^3} = \frac{4\pi^2\mu}{k} = \text{Constt} \qquad \text{Here constant} = \frac{4\pi^2\mu}{k}$$

This gives the proof of the kepler's 3rd law of the planetry motion.

☆ In case of Sun earth system

The reduced mass $\mu = m_e$ because in the reduced mass goes towards the smaller value and the earth is very very smaller as compare to the sun.

$$\mu = m_e \;,\quad |k| = Gm_e m_s$$

Then Applying 3rd law of kepler's we have

$$\frac{T^2}{a^3} = \frac{4\pi^2\mu}{k} = \frac{4\pi^2 m_e}{Gm_e m_s} = \frac{4\pi^2}{Gm_s}$$

$$\Rightarrow \quad \boxed{m_s = \frac{4\pi^2}{G}\frac{a^3}{T^3}}$$

See Problem of this chapter

Thus if the period of revolution and semi major axis are given then we can find the mass of the sun such type of question (problem) can be asked in the examination

m_s = Mass of sun —, m_e = mass of earth.

(HOME ASSIGNMENT)

PROBLEM: Equation of the orbit of the particle under the action of central forces are given by

Q.NO 12

TAKWALE

155

(i) $r = a(1+\cos\theta)$

(ii) $r = ae^{b\theta}$

(iii) $\frac{1}{r} = a\cosh b(\theta - \theta_0)$

Where "a" and "b" are constant. Find the corresponding force

SOLUTION:

(I) $r = a(1+\cos\theta)$ $F_{(r)} = ?$

Sol: Diff above w.r.t t we have

$$\frac{dr}{dt} = \frac{d}{dt}\{a(1+\cos\theta)\}$$

$$\frac{dr}{dt} = a\frac{d}{dt}(1+\cos\theta)$$

$$\dot{r} = \frac{dr}{dt} = a\left\{\frac{d}{dt}(1) + \frac{d}{dt}(\cos\theta)\right\}$$

$$\dot{r} = a\left\{0 - \sin\theta\frac{d\theta}{dt}\right\}$$

or $\dot{r} = a(-\sin\theta\,\dot{\theta})$

$$\dot{r} = -a\sin\theta\,\dot{\theta}$$

Again differentiating w.r.t t we have

$$\frac{d}{dt}\dot{r} = -a\frac{d}{dt}(\sin\theta\,\dot{\theta})$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = -a\left[\sin\theta\frac{d\dot{\theta}}{dt} + \dot{\theta}\frac{d}{dt}\sin\theta\right]$$

or $\ddot{r} = -a\{\sin\theta\,\ddot{\theta} + \dot{\theta}\cos\theta\,\dot{\theta}\}$

$$\ddot{r} = -a\sin\theta\,\ddot{\theta} - a\dot{\theta}^2\cos\theta \quad ①$$

multiplying eq ① by μ we get

$$\mu\ddot{r} = -\mu a\sin\theta\,\ddot{\theta} - \mu a\dot{\theta}^2\cos\theta$$

As we know that $F_{(r)} = \mu\ddot{r}$ Then above will be

$$F_{(r)} = \mu\ddot{r} = -\mu a\sin\theta\ddot{\theta} - \mu a\dot{\theta}^2\cos\theta$$

$$\Rightarrow \boxed{F_{(r)} = -\mu a(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)}$$ Ans

(II) If $r = ae^{b\theta}$ Then $F_{(r)} = ?$

Sol: Since $r = ae^{b\theta}$

Differentiate w.r.t t we have

$$\frac{dr}{dt} = \frac{d}{dt}(ae^{b\theta})$$

$$= a\frac{d}{dt}e^{b\theta}$$

$$= ae^{b\theta}\frac{d}{dt}(b\theta) = ae^{b\theta}b\frac{d\theta}{dt}$$

or $\frac{dr}{dt} = ae^{b\theta}\cdot b\dot{\theta}$ $\left(\because \frac{d\theta}{dt} = \dot{\theta}\right)$

or $\dot{r} = \frac{dr}{dt} = \cancel{a}b\dot{\theta}r$ As $r = ae^{b\theta}$

$$\dot{r} = rb\dot{\theta}$$

Again differentiate w.r.to t we have

$$\frac{d}{dt}(\dot{r}) = \frac{d}{dt}(rb\dot{\theta})$$

$$= b\left[r\frac{d}{dt}\dot{\theta} + \dot{\theta}\frac{dr}{dt}\right]$$

$$= b[r\ddot{\theta} + \dot{\theta}\dot{r}]$$ As $\dot{r} = rb\dot{\theta}$ Then

$$= b[r\ddot{\theta} + \dot{\theta}(rb\dot{\theta})]$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = b[r\ddot{\theta} + \dot{\theta}^2 rb]$$

10-05-06

$$\dot{r} = \frac{dr}{dt} = rb(\ddot{\theta} + b\dot{\theta}^2) \quad \text{—①}$$

multiplying by μ we get

$$\mu\ddot{r} = \mu rb(\ddot{\theta} + b\dot{\theta}^2)$$

As we know that $F_{(r)} = \mu\ddot{r}$ then

$$F_{(r)} = \mu\ddot{r} = \mu rb(\ddot{\theta} + b\dot{\theta}^2)$$

As $r = ae^{b\theta}$ Then

$$\boxed{F_{(r)} = \mu abe^{b\theta}(\ddot{\theta} + b\dot{\theta}^2)}$$

Ans.

(III) $\frac{1}{r} = a\cosh b(\theta - \theta_0)$ $F_{(r)} = ?$

Sol: As $\frac{1}{r} = a\cosh b(\theta - \theta_0)$

$$\Rightarrow r = \frac{1}{a\cosh b(\theta - \theta_0)}$$

$$r = \frac{1}{a}\operatorname{sech} b(\theta - \theta_0)$$

differentiating both sides w.r.t t we have

$$\frac{dr}{dt} = \frac{1}{a}\frac{d}{dt}\left(\operatorname{sech} b(\theta - \theta_0)\right)$$

$$= \frac{1}{a}\left(-\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\, b\dot{\theta}\right)$$

$$\dot{r} = \frac{-b}{a}\left(+\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\dot{\theta}\right)$$

Again differentiating w.r.t t we have

$$\frac{d\dot{r}}{dt} = \ddot{r} = \frac{-b}{a}\frac{d}{dt}\left(\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\dot{\theta}\right)$$

10-05-06

$$\ddot{r} = \frac{-b}{a}\Big[\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\frac{d}{dt}(\dot{\theta}) + \operatorname{sech} b(\theta - \theta_0)\frac{d}{dt}(\tanh b(\theta - \theta_0))\dot{\theta} + \tanh b(\theta - \theta_0)\dot{\theta}\frac{d}{dt}(\operatorname{sech} b(\theta - \theta_0))\Big]$$

$$= \frac{-b}{a}\Big[\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\ddot{\theta} + \operatorname{sech} b(\theta - \theta_0)\operatorname{sech}^2 b(\theta - \theta_0)\dot{\theta}(\dot{\theta}b) + \tanh b(\theta - \theta_0)\dot{\theta}\left(-b\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\dot{\theta}\right)\Big]$$

$$\ddot{r} = \frac{-b}{a}\Big\{\operatorname{sech} b(\theta - \theta_0)\tanh b(\theta - \theta_0)\ddot{\theta} + b\dot{\theta}^2\operatorname{sech}^3 b(\theta - \theta_0) - b\dot{\theta}^2\operatorname{sech} b(\theta - \theta_0)\tanh^2 b(\theta - \theta_0)\Big\}$$

$$\ddot{r} = \frac{-b}{a}\operatorname{sech} b(\theta - \theta_0)\Big\{\tanh b(\theta - \theta_0)\ddot{\theta} + b\dot{\theta}^2\operatorname{sech}^2 b(\theta - \theta_0) - b\dot{\theta}^2\tanh^2 b(\theta - \theta_0)\Big\}$$

multiplying by μ both sides we have

$$F_{(r)} = \mu\ddot{r} = \frac{-b\mu}{r}\operatorname{sech} b(\theta - \theta_0)\Big\{\tanh b(\theta - \theta_0)\ddot{\theta} + b\dot{\theta}^2\operatorname{sech}^2 b(\theta - \theta_0) - \tanh^2 b(\theta - \theta_0)b\dot{\theta}^2\Big\}$$

$$\Rightarrow F_r = \frac{-b\mu}{r}\operatorname{sech} b(\theta - \theta_0)\Big\{\tanh b(\theta - \theta_0)\ddot{\theta} + b\dot{\theta}^2\operatorname{sech}^2 b(\theta - \theta_0) - \tanh^2 b(\theta - \theta_0)b\dot{\theta}^2\Big\}$$

Ans

END OF THE Chapter

Written By

MATI ULLAH SURRANI
Physics Department
UOP

CHAP-7

Collisions of Particles

15-05-06

188 TAKWALE

ELASTIC AND INELASTIC SCATTERING

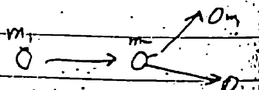
The scattering process can be divided into two catagories Elastic and inelastic scattering.

In Elastic scattering the kinetic energy of all the particles before scattering is equal to that of after scattering. In such a processes the internal nature or the structure of the particles does not change and we can apply the law of conservation of energy without taking into the consideration at all.

In inelastic scattering the Energy and momentenum of the system before and after the collision is not the same. most of collision occurs in nature are inelastic. when two bodies collide at least some amount of heat and sound is produced, thus some part of the initial kinetic energy is lost in this. Therefore the energy and momentum before and after the collision is not the same.

Collision :- when physically impact of two bodies occur then it is called collision.

In the collision the two bodies must be collide with each other.



Scattering :- when the two field interact with each other and as a result of interaction, they scattered at different direction is called scattering.

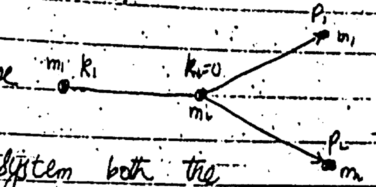
ELASTIC SCATTERING
LABORATORY AND CENTRE OF MASS SYSTEMS

15-05-06

In this system laboratory and center of mass system we use the only elastic scattering. Here in this system we assume that the velocity of the particles are small.

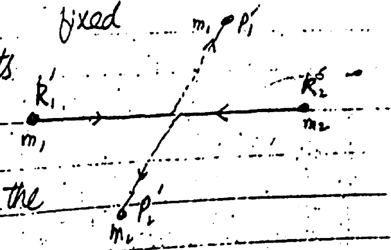
* In laboratory system we consider that one particle is at rest and the other particle is in motion and collide with it.

In this system the m_2 is considered to be at rest.



* In center of mass system both the particles have same velocity and then they collide. In this system the collision between two bodies is treated if they have equal and opposite momenta initially. In this case we also say that 'center of momentum' of colliding particles is fixed.

The actual measurments in experiments are done in the lab system.



If we wish to avail the advantages of the simplifications obtained in a C.m system we must have the transformation relations between the quantities measured in the lab system and those measured in c.m system and vice versa.

15-05-06

Convention: we adopt the following notations here

Quantity	Initial	Final
(1) velocity	u	v
(2) momentum	k	p
(3) K.E	K	T
(4) Scattering angle		θ

The same quantities will be represented in the center of mass system with prime and V will be the velocity of center of mass and $m = m_1 + m_2$

According to law of conservation of linear momentum and K.E, we have

$$k_1 + k_2 = P_1 + P_2 \quad ①$$

$$K_1 + K_2 = T_1 + T_2 \quad ②$$

As we know that $K.E = \frac{1}{2} m V^2 = \frac{m^2 V^2}{2m}$

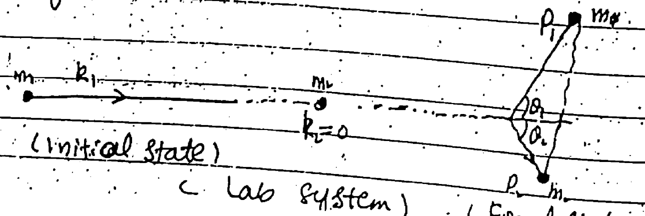
$$K.E = \frac{P^2}{2m}$$

Then eq ② in term of corresponding momenta will be given as

$$\frac{k_1^2}{2m_1} + \frac{k_2^2}{2m_2} = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} \quad ③$$

we have six components of initial momentum we should have some additional information about the problem. It should be about the angle.

Now we describe the Lab. and C.m system seperately.

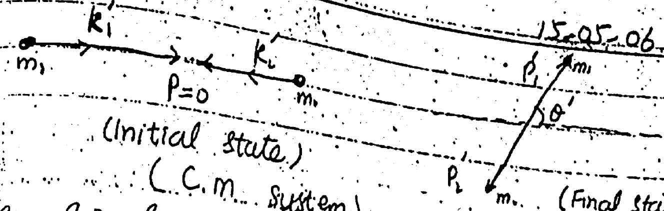


(Initial state)

(Lab system)

(Final state)

15-05-06



(Initial state)

(C.m. System)

(Final state)

The final state in the lab system is redrawn

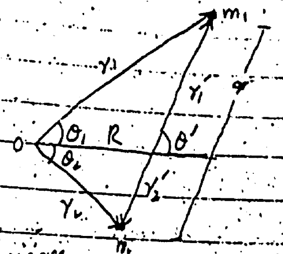


Figure "A"

Let r_1, r_2 and R be the positions vectors of the m_1, m_2 and center of mass respectively.

In the lab system the momentum of the center of mass is

$$P = MV = (m_1 + m_2) V \quad ④$$

Where V is the velocity of the center of mass

In center of mass system r_1' and r_2' represents the positions vectors of m_1 and m_2 then

$$\star \quad r = r_1' + r_2' = r_1 - r_2 \quad ⑤$$

Since the origin of the C.m system is situated at the center of mass then

$$m_1 r_1' + m_2 r_2' = 0 \quad ⑥$$

Add and substract $m_2 r_1'$ to eq ⑥ we have

$$m_1 r_1' + m_2 r_1' + m_2 r_2' - m_2 r_1' = 0$$

$$(m_1 + m_2) r_1' + (m_2 (r_2' - r_1')) = 0$$

As from eq ⑤ $r = r_1' - r_2' \Rightarrow -r = r_2' - r_1'$ putt in above

$$(m_1 + m_2) r_1' + m_2 (-r) = 0$$

$$r_1' (m_1 + m_2) = m_2 r$$

$$r_1' = \frac{m_2}{(m_1 + m_2)} r$$

$$r_1' = \frac{m_1 m_2}{m_1 (m_1 + m_2)} r = \frac{\mu}{m_1} r$$

15-05-06

$$r_1' = \frac{\mu}{m_1} r \quad — (8)$$

reduced mass — where $\mu = \frac{m_1 m_2}{(m_1+m_2)}$

Similarly add and substract $m_1 r_2'$ to eq (8) we have

$$m_1 r_1' + m_1 r_2' + m_2 r_2' - m_1 r_2' = 0$$

$$m_1 r_2' + m_2 r_2' + m_1 r_1' - m_1 r_2' = 0$$

$$r_2'(m_1 + m_2) + m_1(r_1' - r_2') = 0$$

As from eq (5) $r = r_1' - r_2'$ Then

$$r_2'(m_1+m_2) + m_1 r = 0$$

$$r_2'(m_1+m_2) = -m_1 r$$

or $$r_2' = \frac{-m_1}{(m_1+m_2)} r$$

$$r_2' = -\frac{m_1 m_2}{m_2(m_1+m_2)} r = -\frac{\mu}{m_2} r$$

$$\Rightarrow r_2' = -\frac{\mu}{m_2} r \quad — (8)$$ where $\mu = \frac{m_1 m_2}{m_1+m_2}$

differenting eq (7) and eq (8) we have

$$\dot{r}_1' = \frac{\mu}{m_1}\dot{r} \quad \text{and} \quad \dot{r}_2' = -\frac{\mu}{m_2}\dot{r} \quad —— (9)$$

As We know from eq (5)

~~Then~~ But $r = r_1 - r_2 = r_1' - r_2'$

$$\dot{r} = \dot{r}_1 - \dot{r}_2 = \dot{r}_1' - \dot{r}_2' = u \quad — (10)$$

where u is the relative velocity of the 1st particle to the 2nd particle

From eq (9) we have

$$m_1 \dot{r}_1' = \mu \dot{r} \quad \text{and} \quad -m_2 \dot{r}_2' = \mu \dot{r}$$

or $$m_1 \dot{r}_1' = \mu \dot{r} = -m_2 \dot{r}_2' \quad — (11)$$

Thus
equal
In
of

Since

Thus
This
before
magn
In
(Conservation of m
(Conservation of Energ

be

$$\Rightarrow m_1 \dot{r}_1' = -m_2 \dot{r}_2'$$

16-05-06

Thus the linear momentum of the particles are equal and opposite both in initial and final state. In the c.m system the laws of conservation of linear momentum and Kinetic energy is

$$k_1' + k_2' = p_1' + p_2' = 0$$

or $$|k_1'| = |k_2'| \text{ ; and } |p_1'| = |p_2'| \quad — (12)$$

Since from eq (11) we have

$$m_1 \dot{r}_1' = k_1' = \mu \dot{r} = \mu u = -m_2 \dot{r}_2' = -k_2'$$

$$\Rightarrow k_1' = \mu u = -k_2'$$

$$+ \quad p_1' = \mu u = -p_2'$$

As

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \quad — (13)$$

From eq (13) we have

$$\frac{k_1'^2}{2\mu} = \frac{p_1'^2}{2\mu} = \frac{1}{2}\mu u^2 \quad — (14)$$

Thus, we have $u_1' = v_1'$ and $u_2' = v_2'$

This shows That velocities in c.m system before and after collisions are equal in magnitude.

In the Lab system, eq (1) + eq (2) have the form.

(Conservation of Momentum) → $$k_1 = p_1 + p_2 \quad — (14a)$$

(Conservation of Energy) and → $$K_1 = T_1 + T_2 \quad — (14b)$$

and as we have $K.E = \frac{1}{2} m v^2$

Then (14b) can be written as

$$\frac{k_1^2}{2m_1} = \frac{p_1^2}{2m_1} + \frac{p_2'^2}{2m_2} \quad — (15)$$

because 2nd body is at rest in lab system

16-05-06

Relationship between different quantities. In the Lab and C.M System,

we now obtain the relationship between velocities, momenta and scattering angle in the lab and c.m system.

From the figure Ⓐ we have

$$r_1 = R + r_1' \text{ and } r_2 = R + r_2' \quad \text{— ①}$$

Their corresponding velocities are

$$\dot{r}_1 = \dot{R} + \dot{r}_1' \text{ and } \dot{r}_2 = \dot{R} + \dot{r}_2' \quad \text{— ②}$$

As $\dot{r}_1 = v_1$ and $\dot{r}_2 = v_2$ Then above will be

$$\dot{r}_1 = v_1 = \dot{R} + \dot{r}_1' \text{ and } v_2 = \dot{r}_2 = \dot{R} + \dot{r}_2'$$

putting the values of $\dot{r}_1'$ & $\dot{r}_2'$ from Eq ②

$$v_1 = \dot{R} + \frac{\mu}{m_1}\dot{r} \quad \text{and} \quad v_2 = \dot{R} - \frac{\mu}{m_2}\dot{r}$$

As $\dot{R} = V$ and $\dot{r} = u$ then

$$v_1 = V + \frac{\mu}{m_1}u \ , \ v_2 = V - \frac{\mu}{m_2}u \quad \text{—— ②}$$

As we know that

$$P_1 = m_1 v_1 \qquad P_2 = m_2 v_2$$

putting the values of v_1 & v_2 we have

$$P_1 = m_1\left(V + \frac{\mu}{m_1}u\right) \ , \ P_2 = m_2\left(V - \frac{\mu}{m_2}u\right)$$

$$\left.\begin{array}{l} P_1 = m_1 V + \mu u \ , \ P_2 = m_2 V - \mu u \end{array}\right\} \text{— ③}$$

*

MATI ULLAH 0345-9798233
Physics Deptt
University of Peshawar
Jahangir Calozai SURRANI
BANNU

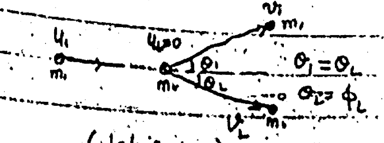
Relat
in
As
we
velocit
the
C.m
Here in a
$\theta_1 = \theta$
$\theta_2 = \phi_L$
In
by
obtai
From
dividi

Relationship between Scattering Angles in Lab (θ_L) and C.m (θ_c) System.

17-05-06

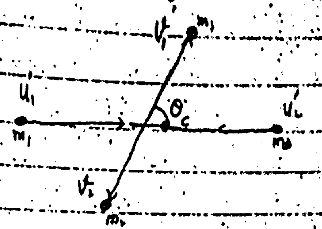
As we have generally

We superimpose (add) the velocities V.


u_1
m_1
$u_2 = 0$
m_2
v_1
m_1
θ_1
θ_L
$\theta_1 = \theta_L$
$\theta_2 = \phi_L$
v_2
m_2

(lab system)

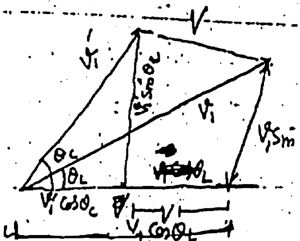
The adding some portion of C.m and Lab system


v_1'
m_1
u_1
m_1
θ_c
v_2'
m_2
v_2'
m_2

(C.m system)

Here in above

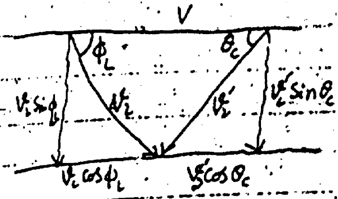
$\theta_1 = \theta_L$
$\theta_2 = \phi_L$


V
v_1'
$v_1' \sin\theta_c$
v_1
$v_1 \sin\theta_L$
θ_c
θ_L
$v_1' \cos\theta_c$
V
u
$v_1 \cos\theta_L$

HARE

$$v_1' + V = v_1$$

(Figure A)


V
ϕ_L
θ_c
$v_2 \sin\phi_L$
v_2
v_2'
$v_2' \sin\theta_c$
$v_2 \cos\phi_L$
$v_2' \cos\theta_c$

(Figure B)

In the velocity representation v_1 is obtained by superposing V on v_1' and v_2 is obtained by superposing V on v_2'.

From the Figure (A) we have

$$\left.\begin{array}{l} v_1' \sin\theta_c = v_1 \sin\theta_L \\ v_1' \cos\theta_c + V = v_1 \cos\theta_L \end{array}\right\} \text{— ①}$$

dividing them on each other

$$\frac{v_1' \sin\theta_c}{v_1' \cos\theta_c + V} = \frac{v_1 \sin\theta_L}{v_1 \cos\theta_L}$$

17-05-06

$$\frac{\sin\theta_L}{\cos\theta_L} = \frac{v_1'\sin\theta_c}{v_1'\left(\cos\theta_c + \frac{V}{v_1'}\right)}$$

$$\Rightarrow \tan\theta_L = \frac{\sin\theta_L}{\cos\theta_L} = \frac{\sin\theta_c}{\left(\cos\theta_c + \frac{V}{v_1'}\right)} \quad ②$$

From The Figure (B) we have

$$v_2\sin\phi_L = v_2'\sin\theta_c \quad ⓐ$$

$$v_2\cos\phi_L + v_2'\cos\theta_c = V$$

$$v_2\cos\phi_L = V - v_2'\cos\theta_c \quad ⓑ$$

Dividing ⓐ on ⓑ we have

$$\frac{\cancel{v_2}\sin\phi_L}{\cancel{v_2}\cos\phi_L} = \frac{v_2'\sin\theta_c}{V - v_2'\cos\theta_c}$$

$$\tan\phi_L = \frac{\cancel{v_2'}\sin\theta_c}{\cancel{v_2'}\left(\frac{V}{v_2'} - \cos\theta_c\right)} = \frac{\sin\theta_c}{\left(\frac{V}{v_2'} - \cos\theta_c\right)} \quad ③$$

As we know that

$$\left.\begin{aligned} V &= \frac{m_1}{(m_1+m_2)}u_1 \\ \text{and}\quad v_1' &= \frac{m_2}{(m_1+m_2)}u_1 = u_1' \\ v_2' &= \frac{m_1}{(m_1+m_2)}u_1 = u_2' \end{aligned}\right\} \quad ④$$

As we know That in c.m System

① The momenta of the Particles are equal and opposite before and after collision

② The motion is Collinear before and after Collision

③ Velocities of Particles before and after Collision are equal.

17-05-06

As $\frac{V}{v_1'}$

putting values of V and v_1' from eq ④

$$\left.\begin{aligned} \frac{V}{v_1'} &= \frac{(m_1/(m_1+m_2))\,u_1}{(m_2/(m_1+m_2))\,u_1} = \frac{m_1}{m_2} \\ \text{and}\quad \frac{V}{v_2'} &= \frac{(m_1/\cancel{(m_1+m_2)})\,u_1}{\{m_1/(m_1+m_2)\}\,u_1} = 1 \end{aligned}\right\} \quad ⑤$$

putting The value of $\frac{V}{v_1'}$ and $\frac{V}{v_2'}$ in eq ② and eq ③ we have

$$\boxed{\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + \frac{m_1}{m_2}}} \quad ⑥$$

① If $m_1 << m_2$ Then $\frac{m_1}{m_2} \approx 0$

$\Rightarrow \tan\theta_L = \tan\theta_c$

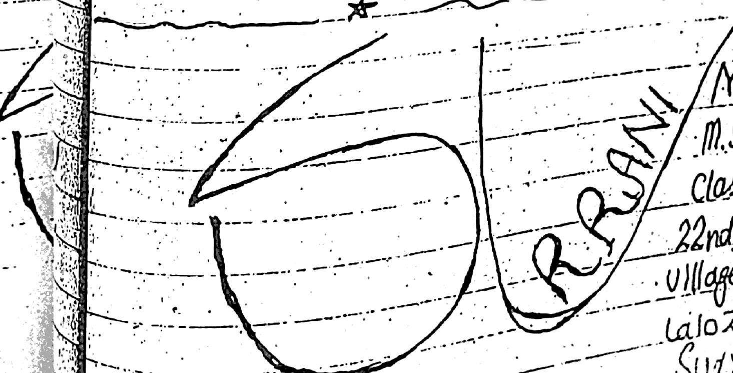
$\theta_L = \theta_c$

② If $m_1 = m_2$ and $\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + 1} = \tan\frac{\theta_c}{2}$ $\boxed{\theta_L = \frac{\theta_c}{2}}$

See imp Result page 116

$$\tan\phi_L = \frac{\sin\theta_c}{(1-\cos\theta_c)} = \frac{2\sin\frac{\theta_c}{2}\cos\frac{\theta_c}{2}}{2\sin^2\frac{\theta_c}{2}}$$

$$\Rightarrow \boxed{\tan\phi_L = \cot\frac{\theta_c}{2}}$$



0345-9798233
MATIULLAH
M.Sc (previous)
Class No. 12
22nd May 2006
Village Jahangir
Lalozai P/o. Torka
Surani Bannu

22-05-06

CROSS-SECTION :-

# 202 TAKWALE

In scattering we usually calculate the "cross section" (σ) and impact parameter (b). The collisions of large particles or classical particles can be observed and the result derived to it can be applied to these. However we could never predict the value of the scattering angle since the nature of the law of force is not known. The Problem of Scattering of micro particles will be dealt in an essentially different way. A beam of such microparticles is made incident on the Target Particles under investigation. The scattered particles at some scattering angle is recorded. Since we are dealing not with a single particle but with beam of incident and scattered particles, we have to speak in Terms of Probability of Scattering of some of The particles in some specified direction. The Probability is expressed in terms of The differential cross section.

In classical mechanics The Trajectories are well defined, while in quntum mechanics there is no well defined Trajectories (because of uncertainty)

If the scatterer have mass 'm'

Then $L = \vec{r} \times p = \frac{m_1}{p} V_\infty b = b\sqrt{2m_1T_0}$ — ①

As $T_0 = \frac{p^2}{2m} \Rightarrow (p = \sqrt{2mT_0})$

As $P = mV_\infty$ by putting value of P in above

Let
① Int
and
Unit
② we
is c
If is
decre
betwe
betwe
Let
Scatt
The
as
∴ di
"b" and
then ①

22-05-06

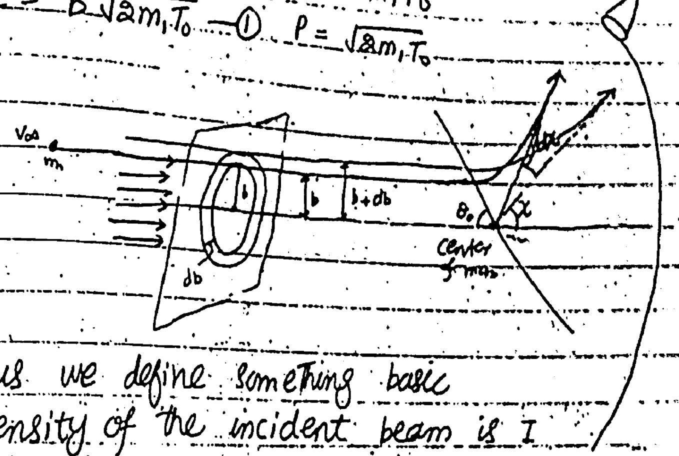
$L = m_1V_\infty b.$

$L = Pb$ As $P = m_1V_\infty$

Putting value of P we get And $T_0 = \frac{P^2}{2m_1}$

$P^2 = 2m_1T_0$

$L = b\sqrt{2m_1T_0}$ — ① $P = \sqrt{2m_1T_0}$



Let us we define something basic

① Intensity of the incident beam is I and is the Number of Particles crossing the unit area held normal to The beam per second

② we consider for a given parameter b, there is a unique angle of scattering χ

It is assumed that $b(\chi)$ is a monotonically decreasing function. Then all the particles between b and b+db will be scattered between χ and $\chi + d\chi$

Let dN is The Number of Particles Scattered by χ and $\chi + d\chi$ per unit area. The differential cross section is defined as $d\sigma = \frac{dN}{I} = \frac{\text{Particles Scattered}}{\text{Incident Particles}}$

∴ dN is the Number of Particles crossing "b" and b+db area ($2\pi b db$)

then since $dN = 2\pi b db (I)$

Then $d\sigma = \frac{dN}{I} = \frac{2\pi b db (I)}{I}$

22-05-06

$$d\sigma = 2\pi b\, db$$

$$\text{or}\quad d\sigma = 2\pi b \left|\frac{db}{d\chi}\right| d\chi \text{ — ⓐ}$$

$$\because\ b(\chi) = b$$

In eq ⓐ $\frac{db}{d\chi}$ is negative (as b increasing Then χ decreases)

Therefore we write absolute $\left|\frac{db}{d\chi}\right|$ otherwise eq ⓐ ⇒

$$d\sigma = -2\pi b \frac{db}{d\chi}\, d\chi \text{ — ⓑ}$$

Usually $d\sigma$ is reffered to solid angle $d\Omega$ rather then a plane angle $d\chi$

The solid angle between the semi verticle angle χ and $\chi + d\chi$ is Here $\theta = \chi$

$$d\Omega = 2\pi \sin\theta\, d\theta \text{ — ⓑ}$$

Then dividing eq ⓑ over ⓑ we have

$$\frac{d\sigma}{d\Omega} = \frac{+2\pi b \left|\frac{db}{d\chi}\right| \cancel{d\chi}}{\cancel{2\pi} \sin\chi\, \cancel{d\chi}}$$

$$\text{or}\quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right| = \frac{-b}{\sin\chi}\frac{db}{d\chi} \text{ — ⓒ}$$

This is the differential Cross section, it has the dimension $[L^2]$

The total cross section is

$$\sigma_{Total} = \int \frac{d\sigma}{d\Omega}\, d\Omega \text{ — ⓓ}$$

This is the last relation which gives us the the total cross section.

* * *

By MATI ULLAH of BANNU
M.Sc (PREVIOUS)
Physics Department UOP

23-05-06

Relation Between Cross Sections in the C.M and Laboratory Systems :-

Total # 208

So far, the problem of scattering of particles was considered in the c.m system. We will consider the transformation of the cross section from the c.m system to the laboratory system. We calculate the cross section in c.m frame but measure it in lab frame.

- The number of particles scattered in θ_l and $\theta_l + d\theta_l$ is same as the number of particles scattered in θ_c and $\theta_c + d\theta_c$.

Thus we have

$$d\Omega_c \left.\frac{d\sigma}{d\Omega}\right|_{c.m} = d\Omega_l \left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

As $d\Omega_c = 2\pi \sin\theta_c\, d\theta_c$ and $d\Omega_l = 2\pi \sin\theta_l\, d\theta_l$

then above will be

$$\cancel{2\pi} \sin\theta_c\, d\theta_c \left.\frac{d\sigma}{d\Omega}\right|_{c.m} = \cancel{2\pi} \sin\theta_l\, d\theta_l \left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$\Rightarrow \sin\theta_c\, d\theta_c \left.\frac{d\sigma}{d\Omega}\right|_{c.m} = \sin\theta_l\, d\theta_l \left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{c.m} = \frac{\sin\theta_l\, d\theta_l}{\sin\theta_c\, d\theta_c} \left.\frac{d\sigma}{d\Omega}\right|_{lab} \text{ — ①}$$

As we know that from previous knowledge

$$\tan\theta_l = \frac{\sin\theta_c}{\cos\theta_c + m_1/m_2} \text{ — Ⓐ}$$

Now we have different cases.

$\boxed{\text{Case 1}}$:- If masses $m_1 = m_2$ Then putting eq in eq Ⓐ we have

$$\tan\theta_l = \frac{\sin\theta_c}{\cos\theta_c + \frac{m_1}{m_1}} = \frac{\sin\theta_c}{\cos\theta_c + 1}$$

23-05-06

$$\tan\theta_\ell = \frac{\sin\theta_c}{\cos\theta_c+1} = \frac{2\sin\theta_c/_2\,\cos\theta_c/_2}{2\cos^2\theta_c/_2}$$

$$\tan\theta_\ell = \frac{\sin\theta_c/_2}{\cos\theta_c/_2} = \tan\theta_c/_2$$

$$\tan\theta_\ell = \tan\theta_c/_2$$

$$\Rightarrow \theta_\ell = \theta_c/_2$$

putting This value in eq ① we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{c.m} = \frac{\sin\theta_\ell\, d\theta_\ell}{\sin\theta_c\, d\theta_c}\left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$= \frac{\sin\frac{\theta_c}{2}\,\frac{d\theta_c}{2}}{\sin\theta_c\, d\theta_c}\left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$= \frac{1}{2}\,\frac{\sin\theta_c/_2\, d\theta_c}{2\sin\theta_c/_2\,\cos\frac{\theta_c}{2}\, d\theta_c}\left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{1}{4}\,\frac{1}{\cos\frac{\theta_c}{2}}\left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$\Rightarrow \left.\frac{d\sigma}{d\Omega}\right|_{lab} = 4\cos\frac{\theta_c}{2}\left.\frac{d\sigma}{d\Omega}\right|_{cm}$$

$$\Rightarrow \boxed{\left.\frac{d\sigma}{d\Omega}\right|_{lab} = 4\cos\theta_\ell\left.\frac{d\sigma}{d\Omega}\right|_{c.m}}$$

Case 2:- If $m_2 \gg m_1$ Than $\frac{m_1}{m_2} \approx 0$

Then eq Ⓐ

$$\tan\theta_\ell = \frac{\sin\theta_c}{\cos\theta_c + \frac{m_1}{m_2}}$$

23-05-06

$$\tan\theta_\ell \approx \frac{\sin\theta_c}{\cos\theta_c + 0} \approx \tan\theta_c$$

$$\Rightarrow \tan\theta_\ell \approx \tan\theta_c$$

$$\Rightarrow \theta_\ell \approx \theta_c$$

Then putting in eq ① we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{c.m} \approx \frac{\sin\theta_\ell\, d\theta_\ell}{\sin\theta_c\, d\theta_c}\left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{c.m} = \frac{\sin\theta_c\, d\theta_c}{\sin\theta_c\, d\theta_c}\left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

$$\Rightarrow \left.\frac{d\sigma}{d\Omega}\right|_{c.m} = \left.\frac{d\sigma}{d\Omega}\right|_{lab}$$

————————————

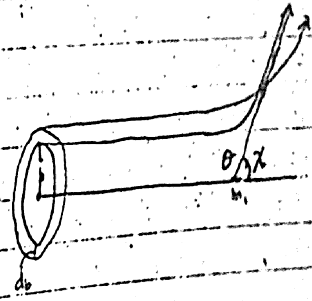
HOME Assignment

PROBLEM:- Find an effective cross section of Scattering of Particles from a heavy rigid sphere of radius R

SOLUTION:-

we know That

$$\sigma_{Total} = \int \frac{d\sigma}{d\Omega}\, d\Omega \quad \text{Ⓐ}$$



we have

$\chi = \pi - \theta$ from the figure then

$$d\chi = -d\theta$$

and $\sin\chi = \sin(\pi-\theta)$

$$= \sin\pi\cos\theta - \cos\pi\sin\theta = \sin\theta$$

$$\sin\chi = -(-1)\sin\theta = \sin\theta$$

and $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right| = \frac{-b}{\sin\chi}\frac{db}{d\chi}$

putting in eq Ⓐ

25-05-08

$$\sigma_{Total} = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int \frac{-b}{\sin\chi}\, \frac{db}{d\chi}\, d\Omega$$

As $\sin\chi = \sin\theta$, $d\chi = -d\theta$

And solid angle $= 2\pi \sin\theta\, d\theta = d\Omega$ then

$$\sigma_{Total} = \int_0^R \frac{-b}{\sin\theta}\, \frac{db}{(-d\theta)}\, 2\pi \sin\theta\, d\theta$$

$$= \int_0^R 2\pi\, b\, db = 2\pi \int_0^R b\, db$$

$$\sigma_{Total} = 2\pi \frac{b^2}{2}\Big|_0^R = 2\pi \frac{R^2}{2} = \pi R^2$$

$$\boxed{\sigma_{Total} = \pi R^2}$$

In Case $\chi = \pi - 2\theta$, we to $d\Omega = 4\pi \sin 2\theta\, d\theta$

$d\Omega = 4\pi \sin 2\theta\, d\theta$

* *

RUTHERFORD SCATTERING

To our knowledge first microscopic meterial particle scattering experiment was done by Lord Rutherford Rutherford deduced the nature of basic interaction from measurement of scattering cross-section. He performed scattering of alpha particles from gold atom in the gold foil. From his observation very few alpha particles get scattered in the back ward direction, he concluded that atomic model is similar to the solar system model.

26-05-08

A heavy nucleus ($\approx 10^{-24}$ gm) is at the center and electrons ($m = \approx 10^{-27}$ gm) revolve around it. Size of of the atom was known to be $\approx 10^{-8}$ cm, where as Rutherford experiment concluded that nuclear size is $\approx 10^{-12}$ cm. Since Alpha particles have charge 2e and charge on gold nucleus is 79e. Therefore particles with zero impact parameter will come closest, where their potential and kinetic energies are equal. Thus

$R \simeq$ distance of closest approach $\simeq$ size of the nucleus.

Thus

$$\frac{1}{4\pi\epsilon_0}\, \frac{2e \cdot 79e}{R} = \text{K.E of } \alpha\text{-particles} = 7.9 \text{ mev}$$

$$\Rightarrow R \simeq \frac{1}{4\pi\epsilon_0}\, \frac{2e \cdot 79e}{7.9 \text{ mev}} \simeq 10^{-12} \text{ cm}$$

This gives the size of nucleus $= 10^{-12}$ cm

Coulomb Scattering Cross-Section:

Coulomb scattering cross section:

we will evaluate scattering cross section where projectile has a charge Ze and charge of Target Ze interacting with Coulomb's interaction. we want to find $b(\chi)$ so that

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi} \left| \frac{db}{d\chi} \right| \quad \text{—} ①$$

can be evaluated.

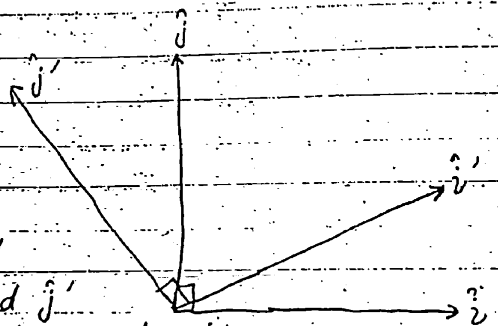
25-05-06

We evaluate $b(\chi)$ by eq (i) using lenz-vector and (ii) by conventional integration.

① USE OF LENZ VECTOR

It can be shown that the lenz-vector is constant for $\frac{1}{r}$ potential.

Thus $\bar{A} = \vec{P} \times \vec{L} + \frac{\mu K \vec{r}}{r}$ — ②



Let $\hat{i}$ be incoming direction and $\hat{j}$ at right angles to it. Let $\hat{i}'$ be the scattered direction and $\hat{j}'$ be perpendicular to it.

For incoming beam $\bar{P} = \bar{P}\hat{i} = \mu v_\infty \hat{i}$

The direction of $\bar{L}$ is $\hat{n}$ and is perpendicular to the plane of the paper.

$$\bar{L} = \mu v_\infty b \hat{n} = \vec{L}'$$

Where $\vec{L}'$ is the angular momentum of the final direction.

Thus the lenz vector for initial state (1) is

$$\bar{A} = (\mu v_\infty)(\mu v_\infty b)(\hat{i} \times \hat{n}) + \mu \frac{K\bar{r}}{r}$$

$$\bar{A} = +(\mu v_\infty)^2 b \hat{j} + \mu K \hat{i} \quad — ③$$

As $\hat{i} \times \hat{n} = +\hat{j}$

25-05-06

Let the lenz vector of final state is

$$\vec{A} = \bar{P}' + \bar{L}' + \mu \frac{K\bar{r}}{r}$$

$$= (\mu v_\infty)(\mu v_\infty b)(\hat{i}' \times \hat{n}') + \mu K \hat{i}'$$

$$\vec{A} = (\mu v_\infty)^2 b \hat{j}' + \mu K \hat{i}' \quad — ④$$

Taking a dot product of $\hat{i}$ with $\bar{A}$ in eq ③ ~~by putting~~ and eq ④ value of v_∞ we have

$$\hat{i} \cdot \bar{A} = -\mu K = 2T \mu b \hat{i} \cdot \hat{j}' + \mu K \hat{i} \cdot \hat{i}' \quad — ⑤$$

$T = \frac{1}{2}\mu v_\infty^2$

$v_\infty = \sqrt{\frac{2T}{\mu}}$

where in eq ⑤ $T = \frac{1}{2}\mu v_\infty^2$ is Kinetic Energy

$\because \hat{i} \cdot \hat{j}' = -\sin\chi$, $\hat{i} \cdot \hat{i}' = \cos\chi$

Thus we get

$$+K(1 + \cos\chi) = +2T b \sin\chi$$

$$\boxed{b = \frac{K}{2T} \cot\frac{\chi}{2}} \quad — ⑥$$

This equation will come it will come in this way.

By putting values of $\hat{i}\hat{j}$ and $\hat{i}\cdot\hat{i}' = \cos\chi$ in eq ⑤ we have

$$-\mu K = 2T\mu b(-\sin\chi) + \mu K \cos\chi$$

$$-\mu K - \mu K \cos\chi = -2T \mu b \sin\chi$$

$$+\mu(K + K\cos\chi) = +2\mu T b \sin\chi$$

$$K(1 + \cos\chi) = 2T b \sin\chi$$

$$b = \frac{K(1 + \cos\chi)}{2T \sin\chi}$$

$$= \frac{K \; 2\cos^2 \chi/2}{2T \; 2\sin\chi/2 \cos\chi/2}$$

$$\boxed{b = \frac{K}{2T} \cot\frac{\chi}{2}} \quad — ⑥$$

25-05-06

The differential cross section is calculated as

$$\frac{db}{d\chi} = \frac{-K}{2T}\frac{1}{2}\operatorname{cosec}^2\chi/2$$

$$\left|\frac{db}{d\chi}\right| = \frac{K}{2T}\frac{1}{2}\operatorname{cosec}^2\chi/2$$

And we know that

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right|$$

putting the values of b and $\left|\frac{db}{d\chi}\right|$ we have

$$\frac{d\sigma}{d\Omega} = \frac{K}{2T}\frac{\cot\chi/2}{\sin\chi}\frac{1}{2}\frac{K}{2T}\operatorname{cosec}^2\frac{\chi}{2}$$

$$= \frac{K^2}{16T^2}\frac{\cancel{\cos}\chi/2}{\cancel{\sin}\chi/2}\frac{1}{2\sin\chi/2\cancel{\cos\chi/2}} \times \frac{1}{\sin^2\chi/2}$$

$$\frac{d\sigma}{d\Omega} = \frac{K^2}{16T^2}\frac{1}{\sin^4\chi/2} \quad — ⑦$$

This with $K = ZZ'e^2$ and $\mu = \frac{m_\alpha + m_{Au}}{m_\alpha + m_{Au}}$.

then we have above equation

$$\frac{d\sigma}{d\Omega} = \frac{(ZZ'e^2)^2}{16T^2}\frac{1}{\sin^4\chi/2}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} \doteq \frac{(ZZ'e^2)^2}{4(2T)^2}\frac{1}{\sin^4\chi/2}} \quad — ⑧$$

This is the Rutherford formula

* *

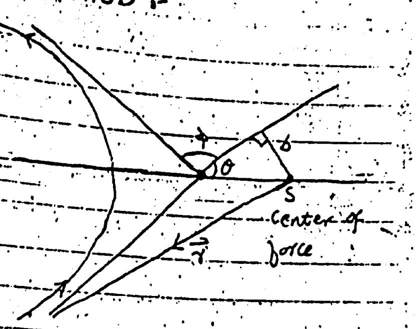
The
Rela
para
Scat
in
Scat

let
alph

If
defi
be
Cen
mo

RUTHERFORD'S SCATTERING & CONVENTIONAL METHOD :-

25-05-06



The figure shows the Relation of orbit parameters and scattering angle in the repulsive scattering.

Let v_0 be the initial velocity of alpha particles then

$$E = \frac{1}{2}\mu v_0^2 \quad — ①$$

If 'b' denotes the impact parameter defined as the perpendicular distance between the incident velocity and center of force. Then the angular momentum is "$\mu v_0 b$"

Since for a central force

$$r^2\theta = l = \text{constant}$$

Then $\mu r^2\theta = \mu l = L$. This denotes the angular momentum.

$$\mu l = \mu v_0 b$$

$$\Rightarrow l = v_0 b = b\sqrt{\frac{2E}{\mu}} \quad — ②$$

$E = \frac{1}{2}\mu v_0^2$

$v_0^2 = \frac{2E}{\mu}$

$v_0 = \sqrt{\frac{2E}{\mu}}$

because from eq ①

$$E = \frac{1}{2}\mu v_0^2$$

$$\Rightarrow v_0^2 = \frac{2E}{\mu}$$

$$\Rightarrow v_0 = \sqrt{\frac{2E}{\mu}}$$

25-05-06

Now the Energy of the System is

$$E = \frac{1}{2}\mu(r^2 + r^2\theta^2) + \frac{ZZ'e^2}{r}$$

$$E = \frac{\mu}{2}\left[r^2 + r^2\theta^2 + \frac{2ZZ'e^2}{\mu r}\right]$$

$$\Rightarrow \frac{2E}{\mu} = r^2 + r^2\theta^2 + \frac{2ZZ'e^2}{\mu r}$$

$$= \left(\frac{dr}{d\theta} \times \frac{d\theta}{dt}\right)^2 + r^2\dot\theta^2 + \frac{2ZZ'e^2}{\mu r}$$

$$= \left(\frac{dr}{d\theta}\right)^2\dot\theta^2 + r^2\dot\theta^2 + \frac{2ZZ'e^2}{\mu r}$$

$$\frac{2E}{\mu} = \dot\theta^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] + \frac{2ZZ'e^2}{\mu r}$$

As we have $r^2\frac{d\theta}{dt} = l \Rightarrow r^2\dot\theta = l$.

$$\Rightarrow \dot\theta = \frac{l}{r^2}$$

Then $\dot\theta^2 = \frac{l^2}{r^4}$

putting this value in above we have

$$\frac{2E}{\mu} = \frac{l^2}{r^4}\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] + \frac{2ZZ'e^2}{\mu r} \quad ③$$

Let $r = \frac{1}{u} \Rightarrow \frac{dr}{d\theta} = \frac{-1}{u^2}\frac{du}{d\theta} \Rightarrow$

putting this value in eq ③ we have

$$\frac{2E}{\mu} = \frac{l^2}{r^4}\left[r^2 + \left(\frac{-1}{u^2}\frac{du}{d\theta}\right)^2\right] + \frac{2ZZ'e^2}{\mu}u$$

25-05-06

$$\frac{2E}{\mu} = l^2u^4\left[\frac{1}{u^2} + \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2\right] + \frac{2ZZ'e^2}{\mu}u$$

$$= \frac{l^2u^4}{u^2} + \frac{l^2u^4}{u^4}\left(\frac{du}{d\theta}\right)^2 + \frac{2ZZ'e^2}{\mu}u$$

$$l^2\left(\frac{du}{d\theta}\right)^2 = \frac{2E}{\mu} - \frac{2ZZ'e^2u}{\mu} - u^2l^2$$

Dividing by l^2 we have

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2E}{\mu l^2} - \frac{2ZZ'e^2}{\mu l^2}u - u^2$$

$$\frac{du}{d\theta} = \sqrt{\frac{2E}{\mu l^2} - \frac{2ZZ'e^2}{\mu l^2}u - u^2}$$

$$\Rightarrow \int d\theta = \int \frac{du}{\sqrt{\frac{2E}{\mu l^2} - \frac{2ZZ'e^2}{\mu l^2}u - u^2}} \quad ④$$

Then using the relation

$$\int \frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{+1}{\sqrt{c}}\cos^{-1}\frac{(cx-b)}{\sqrt{b^2-ac}}$$

The Solution of eq ④ is

As here $a = \frac{2E}{\mu l^2}$, $b = -\frac{ZZ'e^2}{\mu l^2}$, $c = -1$

Then

$$\theta - \theta' = \frac{+1}{\sqrt{1}}\cos^{-1}\frac{(-1)u - \left(-\frac{ZZ'e^2}{\mu l^2}\right)}{\sqrt{\left(\frac{ZZ'e^2}{\mu l^2}\right)^2 - \left(\frac{2E}{\mu l^2}\right)(-1)}}$$

$$\theta - \theta' = \cos^{-1}\frac{\left(u + ZZ'e^2/\mu l^2\right)}{\sqrt{\frac{Z^2Z'^2e^4}{\mu^2l^4} + \frac{2E}{\mu l^2}}}$$

$$\theta = \theta' + \cos^{-1} \frac{u + zz'e^2/\mu l^2}{\frac{zz'e^2}{\mu l^2}\sqrt{1 + \frac{2E}{\mu l^2} \times \frac{\mu^2 l^4}{z^2 z'^2 e^4}}}$$

$$\therefore \theta = \theta' + \cos^{-1} \frac{u + zz'e^2/\mu l^2}{\frac{zz'e^2}{\mu l^2}\sqrt{1 + \frac{2\mu E l^2}{z^2 z'^2 e^4}}}$$

or

$$\theta - \theta' = \cos^{-1} \left\{ \frac{u + zz'e^2/\mu l^2}{\frac{zz'e^2}{\mu l^2}\sqrt{1 + \frac{2\mu E l^2}{z^2 z'^2 e^4}}} \right\}$$

or

$$\cos(\theta - \theta') = \frac{u + zz'e^2/\mu l^2}{\frac{zz'e^2}{\mu l^2}\sqrt{1 + \frac{2\mu E l^2}{z^2 z'^2 e^4}}}$$

$$\cos(\theta - \theta')\sqrt{1 + \frac{2\mu E l^2}{z^2 z'^2 e^4}} = \frac{(u + zz'e^2/\mu l^2)}{zz'e^2/\mu l^2}$$

$$= \frac{\frac{zz'e^2}{\mu l^2}\left(1 + \frac{\mu l^2}{zz'e^2} u\right)}{\frac{zz'e^2}{\mu l^2}}$$

$$\sqrt{1 + \frac{2\mu E l^2}{z^2 z'^2 e^4}} \cos(\theta - \theta') - 1 = \frac{\mu l^2}{zz'e^2} u$$

we choose the coordinate axis such that $\theta' = 0$ i.e.

$$\sqrt{1 + \frac{2E\mu l^2}{z^2 z'^2 e^4}} \cos\theta - 1 = \frac{\mu l^2}{zz'e^4} \frac{1}{r} \quad ⑤ \quad \left(\text{As } u = \frac{1}{r}\right)$$

comparing eq ⑤ with conic section General equation

i-e $\frac{l}{r} = \epsilon\cos\theta - 1$

Where eq ⑤ represent hyperbola

$$\epsilon = \sqrt{1 + \frac{2E\mu l^2}{z^2 z'^2 e^4}} > 1$$

where
$\mu l^2 = 2Eb^2$
$b^2 = \frac{\mu l^2}{2E}$

As $\mu l^2 = 2Eb^2$ then

$$\epsilon = \sqrt{1 + \frac{2E \cdot 2Eb^2}{z^2 z'^2 e^4}} = \sqrt{1 + \left(\frac{2Eb}{zz'e^2}\right)^2} \quad ⑥$$

Hence the center of force lies on the exterior focus of the hyperbola From figure

we have also
$l = \frac{\mu l^2}{zz'e^2}$
Semi latus rectum

$$\phi = \pi - 2\theta$$

$$\theta = \pi/2 - \phi/2$$

From eq ⑤ As $r \to \infty$ Then $\epsilon\cos\theta - 1 = 0$

$$\epsilon \cos\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = 1$$

$$\epsilon\left(\cancel{\cos\pi/2}\cos\phi/2 + \sin\pi/2 \sin\phi/2\right) = 1$$

$$\epsilon \sin\frac{\phi}{2} = 1$$

or $\sin\phi/2 = \frac{1}{\epsilon}$

$\Rightarrow \sin^2\phi/2 = \frac{1}{\epsilon^2}$ As $\sin^2\theta = 1 - \cos^2\theta$

$1 - \cos^2\phi/2 = \frac{1}{\epsilon^2}$

or $\cos^2\phi/2 = 1 - \frac{1}{\epsilon^2} = \frac{\epsilon^2 - 1}{\epsilon^2}$

25-05-06.

$\epsilon^2\cos^2\phi/2 = \epsilon^2-1$ As $\frac{1}{\epsilon^2} = \sin^2\phi/2$

or $\frac{\cos^2\phi/2}{\sin^2\phi/2} = \epsilon^2-1$

$$\cot^2\phi/2 = \epsilon^2-1$$

putting value of ϵ from eq ⑥

$$\cot^2\phi/2 = \left(\sqrt{1+\left(\frac{2Eb}{zz'e^2}\right)^2}\right)^2-1$$

$$\cot^2\phi/2 = \cancel{1}+\left(\frac{2Eb}{zz'e^2}\right)^2 \cancel{-1}$$

Taking underroot on both sids

$$\cot\phi/2 = \frac{2Eb}{zz'e^2}$$

$$\Rightarrow b = \frac{zz'e^2}{2E}\cot\phi/2 \quad ⑦$$

The eq ⑦ gives the impact parameter.
The cross section for equation ⑦ can be Calculated as

$$\sigma(\phi) = \frac{-b}{\sin\phi}\frac{db}{d\phi} = \frac{b}{\sin\phi}\left|\frac{db}{d\phi}\right| \quad ⑧$$

From eq ⑦ we have

$$\frac{db}{d\phi} = -\frac{zz'e^2}{2E}\frac{1}{2}\operatorname{cosec}^2\phi/2$$

or $$\left|\frac{db}{d\phi}\right| = \frac{zz'e^2}{2E}\frac{1}{2}\operatorname{cosec}^2\phi/2 \quad ⑨$$

putting eq ⑨ in eq ⑧ we get

25-05-06

$$\sigma(\phi) = \frac{\frac{zz'e^2}{2E}\cot\phi/2}{\sin\phi} \times \frac{zz'e^2}{4E}\frac{1}{\sin^2\phi/2}$$

$$= \left(\frac{zz'e^2}{2E}\right)^2\frac{\cot\phi/2}{2\sin\phi}\cdot\frac{1}{\sin^2\phi/2}$$

$$= \left(\frac{zz'e^2}{2E}\right)^2\frac{1}{2}\frac{\cancel{\cos\phi/2}}{\sin\phi/2}\cdot\frac{1}{2\sin\phi/2\cancel{\cos\phi/2}}\cdot\frac{1}{\sin^2\phi/2}$$

$$\sigma(\phi) = \left(\frac{zz'e^2}{2E}\right)^2\frac{1}{4\sin^4\phi/2} \quad ⑩$$

Equation ⑩ is Rutherford Scattering cross section in c.m system. This was applied by Rutherford in interpreting his experiments on Scattering of α-particles in gold nuclei.

The classical Rutherford Scattering agree well with the quantum mechanical results.

The Rutherford obtained correct conclusions about the structure of the atom and the size of the nucleus.

★ ★

END OF THIS CHAPTER

MATI ULLAH SURRANI 0345-9798233
class NO 11
BANNU
on 25th May 2006
Room NO:73 Hostel NO:1
UNIVERSITY of Peshawar

CHAPTER-VIII

POISSON'S BRACKETS

331 TAKWACE

Poisson's Brackets are useful representation in which the equation of motion can be written in a symmetric form.

The Poisson's Brackets are found to be useful tool in quantum mechanics and field theory.

Hamiltonian equation of motion gives $\dot{q}_j$ and $\dot{p}_j$ interms of q_j and p_j

If $F(q_i, p_i, t)$ is some general function of dynamical variable (q^5, p^5), then the rate of change of $F(q_i, p_i, t)$ involves the quantity called Poisson's Brackets.

These Brackets have structure very similar to Commutation brackets in quantum mechanics.

The Poisson's Brackets of two function F and G are defined as

$$[F, G]_{q,p} = \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} \right) \text{ — ①}$$

$$[F, G]_{p,q} = \sum_k \left(\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} - \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} \right) \text{ — ②}$$

PROPERTEIS :-

We here write only the properties and then see the proof of these properties from the "Khan Sharif" or from the Notes.

It is also very important

(i) $[F, G] = -[G, F]$

The poisson bracket are anticommutative

31-05-06

The Poisson bracket of a quantity with itself is zero i-e commute

$$[F, F] = [G, G] = 0$$

(ii) The Poisson's Brackets of a dynamical variable F, with a constant c, must vanish i-e

$$[F, c] = [c, F] = 0$$

(iii) The poisson's Brackets obeys distributive law of algebra i.e

$$[F+G, W] = [F, W] + [G, W]$$

(iv) The other property is

$$[FG, W] = [F, W]G + F[G, W]$$

(v) The important property of the poisson's bracket is

$$[q_j, p_k] = \delta_{jk}$$

Where δ_{jk} is Kronecker's delta

* *

EQUATION OF MOTION OF POISSON BRACKETS

As we know that

$$[F, G]_{q,p} = \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} \right) \text{ — ①}$$

If we take

$$[q_j, H] = \sum_k \left(\frac{\partial q_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial H}{\partial q_k} \right) -$$

If we take $j = k$ then

$$[q_j, H] = \sum_k \left(\frac{\partial q_k}{\partial q_k}\frac{\partial H}{\partial p_k} - \cancel{\frac{\partial q_k}{\partial p_k}}^{0}\frac{\partial H}{\partial q_k} \right)$$

$$[q_j, H] = \sum_k \left(\frac{\partial H}{\partial p_j} \right) = \frac{\partial H}{\partial p_j} = \dot{q}_j$$

$$\Rightarrow [q_j, H] = \dot{q}_j \text{ — ②}$$

Similarly we have

$$[p_j, H] = \sum_k \left(\frac{\partial p_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_j}{\partial p_k}\frac{\partial H}{\partial q_k} \right)$$

31-05-06

If we have $k = j$ Then

$$[p_j, H] = \sum_k \left(\frac{\partial p_k}{\partial q_k}^{0} \frac{\partial H}{\partial q_j} - \frac{\partial p_k}{\partial p_k} \frac{\partial H}{\partial q_j} \right)$$

$$[p_j, H] = -\frac{\partial H}{\partial q_j} = \dot{p}_j$$

$$\Rightarrow \quad [p_j, H] = \dot{p}_j \quad — ③$$

The eq ② and eq ③ are perfectly symmetric because difference of a negative sign, which is present in Hamiltonian equation is eliminated in Poisson's Brackets

If we write the equations Then in Term of Poisson's Brackets we have

eq ② & ③:

$$\frac{\partial H}{\partial p_j} = \dot{q}_j = [q_j, H]$$

$$-\frac{\partial H}{\partial q_j} = \dot{p}_j = [p_j, H]$$

This is Equation of motion in the Poisson Brackets.

Now we Consider a general function

$$F = F(q_i, p_i, t)$$

Then

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial q_i}{\partial t} + \frac{\partial F}{\partial p_i} \frac{\partial p_i}{\partial t} \right) + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) + \frac{\partial F}{\partial t}$$

putting the values of $\dot{q}_i$ and $\dot{p}_i$ from eq ② + ③ we have

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial F}{\partial t}$$

31-05-06

$$\Rightarrow \quad \frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t} \quad — ④$$

As

$$[F, H] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}$$

Equation of motion in terms of Poisson's brackets:

If F does not contain time explicitly then in eq ④ $\frac{\partial F}{\partial t} = 0$ then the eq ④ can be written as

$$\frac{dF}{dt} = [F, H]$$

If $F = H$ then

$$\frac{dH}{dt} = [H, H] = 0$$

If H does not contain time explicitly then H is a constant of motion The above equation will be

$$\frac{dH}{dt} = 0$$

$$dH = 0 \cdot dt = 0$$

$$dH = 0$$

$$\int dH = \int 0$$

$$\Rightarrow \quad H = \text{Constt}$$

Total Energy of the System $= H =$ Constt

Thus all the quantities having zero value of the Poisson's Brackets with H are Therefore Constant of motion.

31-05-06

END OF Classical MECHANICS

LAST CLASS ON
30th May 2006
ON TEUSDAY

NOTES Written By

Mr. MATI ULLAH 0345-9798233
M.Sc (prevouss)
Physics Deparatment
UNIVERSITY OF Peshawar
Class No. 11 (Morning)
Room No. 73 Hostel No. 1
Peshawar UNIVERSITY
Village Jahangir Lalozai
P/O Torka Surrani
Bannu.

اور بھی غم ہیں زمانے میں محبت کے سوا
راحتیں اور بھی ہیں وصل کی راحت کے سوا

$$\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + \frac{V}{V_1'}}$$

As $\frac{V}{V_1'} = \frac{m_1}{m_2}$ Then

$$\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + \frac{m_1}{m_2}} \quad \text{—} Ⓐ$$

Now we have two cases

Case ① If $m_1 << m_2$ then $\frac{m_1}{m_2} \approx 0$

then

eqⒶ ⇒ $\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + 0} = \tan\theta_c$

$\cancel{\tan}\theta_L = \cancel{\tan}\theta_c$

$$\Rightarrow \boxed{\theta_L = \theta_c}$$

Case ②

If we have

The $m_1 = m_2$ Then $\frac{m_1}{m_2} = 1$

eqⒶ ⇒ $\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + 1} = \frac{2\sin\frac{\theta_c}{2}\cos\frac{\theta_c}{2}}{\cos\left(\frac{\theta_c}{2} + \frac{\theta_c}{2}\right) + 1}$

$$= \frac{2\sin\theta_c/2 \cos\theta_c/2}{\cos\theta_c/2 \cos\theta_c/2 - \sin\theta_c/2 \sin\theta_c/2 + 1}$$

$$= \frac{2\sin\theta_c/2 \cos\theta_c/2}{\cos^2\theta_c/2 - \sin^2\theta_c/2 + 1}$$

$$= \frac{2\sin\theta_c/2 \cos\theta_c/2}{1 - \sin^2\theta_c/2 - \sin^2\theta_c/2 + 1}$$

$$\tan\theta_L = \frac{2\sin\theta_c/2 \cos\theta_c/2}{2 - 2\sin^2\frac{\theta_c}{2}} = \frac{\cancel{2}\sin\theta_c/2 \cos\theta_c/2}{\cancel{2}(1 - \sin^2\theta_c/2)}$$

$$= \frac{\sin\theta_c/2 \cancel{\cos\theta_c/2}}{\cos^{\cancel{2}}\theta_c/2} = \tan\frac{\theta_c}{2}$$

$\cancel{\tan}\theta_L = \cancel{\tan}\frac{\theta_c}{2}$

$$\Rightarrow \boxed{\theta_L = \frac{\theta_c}{2}} \text{ prov}$$

Topics Needed

- ★ Areal velocity
- ★ Relativistic mechanics
- ★ Quntum mechanics
- ★ Statistical mechanics
- ★ Gen momementum
- ★ Hamiltonio

عوامی فوٹو سٹیٹ
کاپی شاپ بازار

231

23-07-06

SOLUTIONS OF PAPERS

From 2006 upto 2009 Annual

Q NO. 1 (a) As given as the force is Conservative Then we know That

$$E = V + T = T + V$$

where E is The Total Energy

Then

$$E = T + V$$

$$T = E - V$$

As given particle moves along x-axis therefore $V = V(x)$ then

$$T = E - V(x)$$

$$\frac{1}{2} m \dot{x}^2 = E - V(x)$$

$$m\dot{x}^2 = 2(E - V(x))$$

$$\dot{x}^2 = \frac{2}{m}(E - V(x))$$

$$\dot{x} = \sqrt{\frac{2}{m}(E - V(x))}$$

As we have $\dot{x} = \frac{dx}{dt}$ then

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - V(x))}$$

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}}$$

Integrating between t_1 to t_2 and x_1 to x_2 we have

24-07-08

$$\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dx}{[\frac{2}{m}(E-V_{(x)})]^{1/2}}$$

$$\boxed{t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E-V_{(x)})}}}$$

proved Required condition

* *

①(b) $\frac{d}{dt}(mT) = F.P$

when The m is not constant write relations when the m is ~~not~~ constant.

Sol:- we know that

$F = P$

A we know $P = mV$

$P = $ ~~$m\frac{dv}{dt}$~~

$\boxed{F = ~~m\frac{dv}{dt}~~}$ A

~~Putting v = bd~~

$F = \frac{d}{dt}(mV)$

$F.P = mV\frac{d}{dt}(mV)$ — ①

As we know That

$\frac{d}{dt}(mV)mV = mV\frac{d}{dt}(mV) + mV.\frac{d(mV)}{dt}$

$\frac{d}{dt}(m^2V^2) = 2mV\frac{d}{dt}(mV)$

24-7-06

⇒ $\frac{1}{2}\frac{dm^2V^2}{dt} = mV.\frac{d}{dt}mV$

⇒ $mV.\frac{dV}{dt} = \frac{1}{2}\frac{d}{dt}(m^2V^2)$

putting in eq ① we have

$F.P = \frac{1}{2}\frac{d}{dt}(m^2V^2) = \frac{d}{dt}m(\frac{1}{2}mV^2)$

$F.P = \frac{d}{dt}(mT)$ where $T = \frac{1}{2}mV^2$

when m is Constant Then we have

$F = P = m\frac{dV}{dt}$ As m = Const

$F = m\frac{dV}{dt}$

$F.P = mV.m\frac{dV}{dt}$

$F.P = m^2(V.\frac{dV}{dt})$ — ②

A we know, that

$\frac{d}{dt}(V.V) = V.\frac{dV}{dt} + V.\frac{dV}{dt}$

$\frac{d}{dt}V^2 = 2V.\frac{dV}{dt}$

⇒ $V.\frac{dV}{dt} = \frac{1}{2}\frac{dV^2}{dt}$

putting This value in eq② we have

$F.P = m^2\frac{1}{2}\frac{dV^2}{dt}$

$= m\frac{d}{dt}\frac{1}{2}mV^2$

$\boxed{F.P = m\frac{dT}{dt}}$ proved

CLASSICAL MECHANICS
PAPER ANNUAL EXAM 2006

① (a) A particle has total energy E and the force on it is due to potential field V(x), show that the time taken by the particle to go from x_1 to x_2

see khan sharif # 46 $t_2 - t_1 = \int_{x_1}^{x_2} \left(\frac{2}{m}(E - V(x))\right)^{-1/2} dx$

For one dimensional motion.

(b) Show that the equation of motion of a particle implies $\frac{d}{dt}(mT) = F \cdot P$

How this equation will change if mass 'm' is constant

② (a) Explain why the number of constraints equation K are such that $3N - K > 0$ for N-particle system.

(b) State different types of constraints giving one example of each.

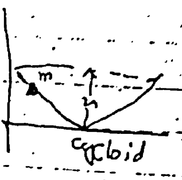
③ A bead slide without friction on a frictionless wire in the shape of cycloid with equation

$$x = a(\theta - \sin\theta)$$
$$y = a(1 + \cos\theta) \quad \text{when } 0 \le \theta \le \pi$$

Find

(a) The Lagrangian function.

(b) The Equation of motion



④ (a) A particle falls vertically under the action of gravity and friction force is obtainable from the dissipation function $G(v) = \frac{1}{2}kv$ Find equation of motion

---

$\frac{\partial L}{\partial \dot r} = P_r$ $\frac{\partial L}{\partial \dot\theta} = P_\theta$ $\frac{\partial L}{\partial \dot\phi} = P_\phi$

31-05-06

(b) Given the Lagrangian $L(r, \theta, \phi) = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2) - V(r)$ Find P_r, P_θ and P_ϕ, which coordinate is cyclic and what are constant of motion for this system?

⑤ Given the velocity dependent central force

$$F = \frac{1}{r^2}\left(1 - \frac{\dot r^2 - 2r\ddot r}{c^2}\right)$$

Write Hamiltonian and Hamiltonian equation of motion?

Q NO ⑥ In general, curves that gives a shortest distance between two points on a surface are called geodesics of the surface. Show that geodesics of a sphere is a great circle.

⑦ Earth (or for that matter any planet) moves around the Sun in elliptic orbit. Prove it. Also obtain essentricity and semi latus rectum of the orbit in terms of constant energy E and angular momentum L.

⑧ Consider a scattering by hard sphere of radius R, where the mass of the sphere is same as that of projectile, then show that

$$\left.\frac{d\sigma}{d\Omega}\right|_{lab} = R^2\cos\theta_2$$

⑨ (a) What is legender transform, show that Lagrangian is Legender transform of the Hamiltonian.

(b) Prove that the coordinate which is cyclic in Hamiltonian is also cyclic in Lagrangian and vice versa.

⑩ (a) Using Poisson Bracket how will you prove that for a conservative system H should be a function of t?

P.T.O

Reference: No. 381-41/Acad-II   Dated: 15-08-2005

Classical Mechanics (Paper III)   Marks: 50

Elementary principles: Brief survey of Newtonian mechanics of a single particle a system of particles, Constraints, D' Alembert's Principle, Lagrange's equations conservative and non-conservative systems and their applications.

Variational Principles: Calculus of variation and Hamilton's Principle, applications Variation principle. Derivation of Lagrange's equations from Hamilton's principle.

Two-Body Central-Force Problem: Two body problem and its reduction to one boc equivalent problem, equation of motion and solution for one body problem, Kepler's law laboratory and center of mass coordinate systems and their mutual transformation Rutherford Scattering formula.

Hamilton's Equation of motion: Hamiltonian functions, Legendre transformation a Hamilton's equation of motion, conservation theorems, cyclic coordinates, Physic significance of Hamiltonian for simple cases.

Canonical Transformations. Examples of canonical transformations, Legrange . an Passion's brackets, Condition for transformation to Canonical Legendre equation of motio Lioville's theorem

✓

1. Goldstien, H.; 1980: Classical Mechanics 2nd edition. Addison Wesley. Readin Massachusetts.
2. Arnold, V.I.; 1980: Mathematical Methods of Clssical Mechanics. Springer Verla New York.
3. Rasband, S.N.; 1983: Dynamics. John Wiley and Son. New York.
4. Matzner, R. A.; 1991: Classical Mechnaics. Prentice Hall, London.
5. Woodhuse, N M. J.; 1087: Introduction to Analytical Dynamics. Oxford Scienc Publications., Oxford.

⑯ (b) Show That

$$\frac{d}{dt}[F, G] = (\frac{dF}{dt}, G] + (F, \frac{dG}{dt})$$

*

*

Written By MATI ULLAH SURRANI 0345 979
PHYSICS Dept
University of Peshawar
Class NO 11

239

University of Peshawar

MA/MSc. Annual Examination, 2000
Physics (PREVIOUS)
Paper-III: Classical Mechinics

Time: 4 Hours Max. Marks: 50
Tuesday, November 21, 2000

NOTE: Attempt ANY FIVE questions. All questions carry equal marks.

Q.1. a. If a particle is moving under the conservative force. Prove that sum of its K.E and P.E remains constant. 5

b. Find an expression for the angular momentum of a system of particle in the center of mass system. 5

Q.2. a. Derive Hamilton's equation of motion. 5

b. Prove that the Hamiltonian or total energy is a constant of motion 5

Q.3. a. Prove that the Poisson's brackets are invariant under canomical transformation. 5

b. Express the Hemilton's equation of motion in term of Poisson's bracket. 5

Q.4. a. Derive Lagrange's equation from D. Alemberts principle. 6

b: Set up the Lagrange's equation for simple pendulum. 4

Q.5. a. Derive Legendre's transformation and obtain Hamilton's equations. 5

b. Show that the transformation 5

$P = \frac{1}{2}(p^2+q^2)$, $Q = \tan^{-1}(q/p)$ is canonical.

Q.6. Discuss the mutual transformation of laboratory and centre of mass co-ordinates system and show that $\theta_L = \theta c/2$ 10

Q.7. a. Prove that for a system of particles the angular momentum and linear momentum remain constant. 6

b. What are Constraints Forces? How are they classified? 4

Q.8. A double pendulum vibrates in a vertical plan. Write the Lagrangian for it and obtain the equation of motion. 10

Q.9. a. Define central force. Give examples of central force in physics. 5

b. Two particles move about each other in circular orbits under the influence of gravitational force with period T. their motion is stopped and they are allowed to fall into each other. Prove that they collide after a time $T/4\sqrt{2}$ 5

Q.10. Write short note on ANY TWO of the following:- 5.5

a. Variational Principle
b. Lagrang's Brackets
c. Normal modes

****

[illegible]

فون: 5703445
عوامی فوٹو سٹیٹ
میٹرک، ایف اے، ایف ایس سی
بی اے، بی ایس سی، ایم۔اے، ایم ایس سی، کے نوٹس دستیاب ہیں۔
محمد علی مارکیٹ کافی شاپ بازار پشاور یونیورسٹی
پروپرائیٹر: جان صاحب
0321-9103972
0345-9223209

240

M.A./M.Sc. Annual Examination, 2001
Physics (Previous)

Paper III: Classical Mechanics

Tuesday, October 2, 2001

Time: 4 Hours, Marks: ~~100~~ 50

Note: Attempt any FIVE questions. All questions carry equal marks.

Q.I. a) Prove that the angular momentum of a particle is constant if no torque is acting.

b) Show that the angular momentum about a point O is the angular momentum of the system concentrated at the centre of mass, plus the angular momentum of motion about the centre of mass.

Q.II. a) What is meant by constraints? Give examples.

b) Discuss in detail the holonomic and non-holonomic constraints.

Q.III. a) Define real and virtual displacement. Derive an equation for the principle of virtual work.

b) Derive D' Alembert's principle.

Q.IV. Derive Langranges equation for a conservative system.

Q.V. Obtain the Lagrangian and equations of motion for the double pendulum, where the lengths of the pendulums are ℓ_1 and ℓ_2 with corresponding masses m_1 and m_2.

Q.VI. A particle of mass m is moving in x – y plane under the influence of a central force:

i) Set up Hamiltonian for the system.

ii) Write Hamilton's equations of motion.

Q.VII. a) What is Hamilton's principle.

b) Derive Hamilton's equations from Hamilton's Principle.

Q.VIII. Derive Rutherford's scattering formula or a particle approaching a heavy nucleus.

Q.IX. State and prove the Keplers laws of plannetary motion.

Q.X. a) What are cyclic coordinates?

b) Show that he transformation:

$P = \frac{1}{2}(p^2 + q^2)$

$Q = \tan^{-1}(q/p)$ is canonical.

****

فون: 5703445
عوامی فوٹو سٹیٹ
میٹرک، ایف اے، ایف ایس سی
بی اے، بی ایس سی، ایم۔ اے، ایم ایس سی، کے نوٹس دستیاب ہیں۔
محمد علی مارکیٹ کافی شاپ بازار پشاور یونیورسٹی
0321-9103972
0345-9223209
پروپرائیٹر: جان صاحب

241

Q.7. (a). Derive the Hamilton's canonical equations.

(b). What are cyclic co-ordinates? Show that the generalized momentum conjugate to a cyclic co-ordinate is conserved. (06) (04)

Q.8. (a). If a system is conservative and independent of time then show that H=E=T+V where H is the total energy of the system. (05)

(b). Using the Hamiltonian and Hamilton's canonical equations, setup and equation of motion for simple pendulum. (05)

Q.9. (a). Define the Poisson brackets and prove that

(i). $[u,v] = -[v,u]$

(ii). $[u,vw] = [u,v]w + v[u,w]$ (06)

(b). Show that the Hamilton's canonical equations can be written in terms Poisson brackets. (04)

Q.10. Write the short notes on any Two of the following: (5,5)

(a) Kinematics in plane polar co-ordinates

(b) Laboratory and center of mass co-ordinate systems

(c) Lagrange's brackets

فون: 5703445
عوامی فوٹو سٹیٹ
میٹرک، ایف اے، ایف ایس سی
بی اے، بی ایس سی، ایم۔ اے، ایم ایس سی، کے نوٹس دستیاب ہیں۔
محمد علی مارکیٹ کافی شاپ بازار پشاور یونیورسٹی
0321-9103972
0345-9223209
پروپرائیٹر: جان صاحب

Salman Ahmad.
B. Sc (Hons) IInd year
University of Peshawar

M.A./M.Sc. Annual Examination-2002
Physics (Previous)
Paper – III: Classical Mechanics

Time Allowed: 4 Hours   Maximum Marks: 50

Note:- Attempt any *FIVE* questions. All questions carry equal marks.

Q.1. (a). Show that if the forces acting on a particle are conservative then the total energy of the particle T+V is conserved. Also state the physical significance of the necessary and sufficient condition on the basis of which you derive the above-mentioned theorem. (05)

(b). Show that the Kinetic Energy of a system can be expressed as the sum of kinetic energy of motion of the center of mass and Kinetic energy of the motion about the center of mass. (05)

Q.2. (a) What is meant by Degree of freedom. (02)

(b). Discuss in detail the various types of constraints with suitable examples (06)

(c) State what are the difficulties that are introduced by the introduction of constraints in the solution of mechanical problems. (02)

Q.3. (a). Starting from D'Alemberts principle derive the Lagrange's equation for a conservative and holonomic system. (08)

(b). What are the advantages of Lagrange's formulation over Newtonian Mechanics. (02)

Q.4. (a). What is Atwood's machine? Write down the equation of motion for it using Lagrange's formulation. (05)

(b). Using the following result of variational principle i.e.

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

Show that the shortest distance between two points is an extremum of a particle in a plane. (05)

Q.5. (a). Define central force, bounded and unbounded motion. (03)

(b). Reduce the problem of motion of two particles of masses m_1 and m_2 located at positions r_1 and r_2 with respect to a fixed origin to an equivalent one body problem. (07)

Q.6. Derive the differential form of equation of the orbit of a particle moving under the influence of a central force also show by using the inverse square law in central force field that equation of orbit satisfies equation of the conic section. (10)

---

243   Classical Mechanics Paper II

M.A./M.Sc. Annual Examination – 2003
Physics (Previous)
Paper – III : Classical Mechanics

Time Allowed: 4 hours   Maximum Marks: 50

Note:- Attempt any *FIVE* questions. All questions carry equal marks.

Q.I. a) Differentiate between Classical Mechanics, Relativistic Mechanics, Quantum Mechanics and Statistical Mechanics. (06)

b) If a Force displaces a particle from point '1' to another point '2', show that the Work done is equal to the change in Kinetic Energy of the particle. (04)

Q.II. a) If the Centre of Mass of a System of many particles is moving with a velocity V with respect to origin, show that total Angular Momentum of the System is equal to the Angular Momentum of the System concentrated at the centre of mass plus Angular Momentum of the System about Centre of Mass. (07)

b) What types of difficulties are introduced in the solution of mechanical problems by involving Constraints in the System? (03)

Q.III. a) State and explain the Principle of Virtual Work. (04)

b) By applying Lagrange's Equations, find out the Radial and Transverse Components of Force acting on a particle in Plane Polar Coordinates (06)

Q.IV. a) State and explain Hamilton's Principle. (04)

b) Define Hamiltonian 'H' and give its physical significance. (06)

Q.V. Discuss the scattering of two particles of masses m_1 and m_2 in the Laboratory and Centre of Mass Coordinate Systems and prove that the scattering angle θ_c in the Centre of Mass System is twice the angle θ_L in the Laboratory System (10)

Q.VI. Consider a particle moving under the influence of a Central Force and show that the Equation for the path of the particle in terms of Plane Polar Coordinates (r, θ) is

$$\theta-\theta_0=\int_{r_0}^{r}\frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left[E-\frac{l^2}{2mr^2}-V(r)\right]}}$$

where l = Angular Momentum, E = Total Energy and $V(r)$ = Potential Energy. (10)

Q.VII. a) State Kepler's Laws of planetary motion. (03)

b) Define Areal Velocity and prove that Areal Velocity of a planet moving under the influence of a Central Force is constant. (07)

Q.VIII. With the help of neat and clean diagrams discuss in detail the Rutherford Scattering of α-particles. (10)

Q.IX. a) Define Poisson Brackets and prove that:

i) $[U, U]=0$ and ii) $[aU+bV, W]=a[U, W]+b[V, W]$

where a and b are constant and U, V and W are functions of position and momentum. (05)

b) Like Hamiltonian 'H', if U is also a function of position, momentum and time, show that: $\frac{dU}{dt}=\frac{\partial U}{\partial t}+[U,H]$

Also, if U is a constant of motion then $[U,H]=0$. (05)

Q.X. Write short notes on any TWO of the following: (10)

a). Lagrange's Equation for a Conservative and Holonomic System

b) Hamilton's Equations of Motion under Canonical Transformations

Silman Ahmad
B.Sc (Hons)-II
Department of Physics
Peshawar University

M.Sc. Annual Examination - 2004
Physics (Previous)
Paper – III : Classical Mechanics

Time Allowed: 4 hours | Maximum Marks: 50

Note: Attempt any *FIVE* questions. All questions carry equal marks.

Q.I. State and explain Hamilton's Variational Principle. Derive Lagrange's Equation from this principles. (10)

Q.II. a) Briefly explain the following terms: (07)
- i) Generalized Momentum
- ii) Hamiltonian and Hamilton's Equation.
- iii) Cyclic Coordinates and General Conservation Theorem.

b) A circuit containing inductance L and capacitance C has its energy given by $\frac{1}{2}L\left(\frac{dq}{dt}\right)^2+\frac{1}{2}\frac{q^2}{c}$. Use Lagrange's Equation to show that $L\left(\frac{d^2q}{dt^2}\right)+\frac{q}{c}=0$ holds for the circuit. (03)

Q.III. a) Define Constraints. What are its various types? Explain each of them by giving at least TWO examples of each. (07)

b) Discuss the difficulties introduced by Constraints in the solution of mechanical problems. (03)

Q.IV. State and derive Kepler's Laws of Planetary motion. (10)

Q.V. a) What do you mean by Legendre Transformation and Canonical Transformation? (05)

b) For the Lagrangian $L=\frac{1}{2}\dot{q}^2-q\dot{q}+q^2$, find generalized momentum 'p' and the Hamiltonian 'H'. Show that the Hamiltonian H reduce to $\frac{P^2}{2}-Q^2$ under canonical Transformation $p=P-Q$ and $q=Q$. (05)

Q.VI. If the Centre of Mass 'M' a two body system is moving with a Velocity 'V' with respect to the origin, show that the Kinetic Energy 'E' of the system is $E=\frac{1}{2}MV^2+\frac{1}{2}\mu v^2$, where μ is the reduced mass of the two body system moving with velocity 'v' with respect to the centre of mass. (10)

Q.VII. a) What do you mean by Poisson Bracket and Integral of Motion? If $U=U(p,q)$ and $V=V(p,q)$ are two Integrals of Motion, show that their Poisson Bracket $[U,V]$ is also an Integral of Motion. (06)

b) For the Hamiltonian $H=\frac{1}{2}(p^2+q^2)$, find $[p,H]$ and $[q,H]$ and hence find the values of $\dot{p}$ and $\dot{q}$. (04)

Q.VIII. a) Derive the Equation of Motion of a particle moving under the influence of a Central Force. (05)

b) Consider a system of Two masses m_1 and m_2 connected by an inextensible string of length 'ℓ' passing over a massless and frictionless pulley. Use Lagrange's Equation to find a relation for the Acceleration of the system. (05)

Q.IX. Write short notes on any TWO of the following: (10)
- a) Laboratory and Centre of Mass Coordinates Systems
- b) Principle of Virtual Work and D'alembert Principle
- c) Calculas of Variations and Euler's Equation.

Q.X. A commet describes a parabolic path, the plane of which is the same as that of the earth's orbit, assumed circular. Show that the maximum length of time during which the commet is able to remain inside the earth's orbit is $\frac{2}{3\pi}$ of a year. (10)

****

VI. (a) Find the Poisson bracket of the function $x(t)=\alpha p^2+2\beta pq+\gamma q^2$ with the Hamiltonian $H=ap^2+bq^2+cp+dq+e$ where $\alpha,\beta,\gamma,a,b,c,d,e$ are all constants.

(b) Show that the Poisson brackets are invariant under canonical transfomation.

VII. (a) A satellite moves around the Earth in an orbit, which passes across the poles. The time at which it crosses each parallel of latitude is measured so that the function θ(t) is known. Explain how you will find the perigee, the semi-major axis and the eccentricity of its orbit in terms θ(t) and the value of g at the surface of the Earth. Assume that the Earth is to be a sphere of uniform density.

(b) Consider the family of orbits in a central potential for which the total energy is constant. Show that, if a stable circular orbit exists, the angular momentum associated with this orbit is larger than for any orbit of the family.

VIII. (a) Classify the following as they are Scleronomic or Rhenonomic, Holonomic or non-holonomic.

[illegible]

- (i) Cylinder rolling without slipping down a rough inclined plane.
- (ii) A particle moving on very long frictionless wire, which rotates with constant angular velocity about horizontal axis.
- (iii) A sphere rolling from a top of a fixed sphere.

(b) A particle of mass m is constrained to move on the inner surface of a bowl having shape of paraboloid of revolution the axis of which is vertical. Initially the particle is located at height z_0 with a horizontal velocity of magnitude $\sqrt{2gz_0}$ and with no vertical velocity. Write down the equation of constraint and find the reaction force of the surface upon the particle.

IX. A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is $F=\frac{1}{r^2}\left(1-\frac{\dot{r}^2-2\ddot{r}r}{c^2}\right)$ where r is the distance of the particle to the center of force. Find the generalized potential that will result in such o force;

X. Find out the degrees of freedom and the equation of constraints for the following dynamical systems.
- (i) A double pendulum → 2×1 2−1=2
- (ii) A disc rolling down on an inclined plane 3−2
- (iii) A particle constrained to move on the surface of sphere 3−1
- (iv) A particle moving on the surface of right circular cone ~~6−2~~ 6−2
- (v) A missile on a submarine.

$f=N-K$

$=4-2=2$

$\frac{\partial H}{\partial q}\frac{\partial X}{\partial p}-\frac{\partial X}{\partial q}\frac{\partial H}{\partial p}$

B.Sc (Hons) Part-I /M.Sc Physics (Previous)
Annual Examination 2005
Paper-III Classical Mechanics

Maximum Marks: 50

Time allowed: 4 Hours

Note: Attempt any five questions.

I. (a) Show that the center of gravity coincides with the center of mass when a body is in uniform gravitational field. What will happen when the force field is not uniform?

(b) A particle of mass m lies at the middle of, a hollow tube of length 2b and mass M, the tube, which is closed at both ends, lies on a smooth table. The coefficient of restitution between m and M is c. Assuming that m moves with initial velocity v_o. Find the velocities of m and M after the first impact, the time required for m to arrive back at middle traveling in the original direction and the velocities relative to the center of mass before and after the first collision.

II. (a) A hollow sphere is filled with water and is used as a pendulum bob. If the water is allowed to flow slowly through a hole at the bottom, how will the period change?

(b) A particle of mass m is suspended at the end of two springs f force constants k_1 and k_2. Write down the equation of motion and find the expression for te frequency.

III. A particle of man m moves in one dimension such that it has Lagrangian

$$L=\frac{m^2\dot{x}^4}{12}+m\dot{x}\,V(x)-V^2(x)$$

Where V is some differentiable function of x. Find the equation of motion for x (t) and describe the physical nature of the system on the basis of this equation.

IV. (a) Distinguish between configuration and phase spaces. Draw diagrams for a stone thrown vertically upward in the field of constant gravity in the two spaces.

(b) Set up Hamiltonian function for a particle of mass m and charge q moving freely in an electromagnetic field with a velocity V. Fiend the generalized momenta.

V. Show that the Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j \text{ can also be written as } \frac{\partial \dot{T}}{\partial \dot{q}_j}-2\frac{\partial T}{\partial q_j}=Q_j$$

$\frac{1}{2}m$

$\frac{\partial}{\partial \dot{q}_j}\frac{dT}{dt} - \partial T/\partial q = Q_j$

$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \partial T/\partial q = Q_j$

[illegible] ∂ 800

P.T.O

$2\frac{\partial}{\partial q_j}(\partial T)$

$T = \frac{1}{2} m \dot{r}^2$

$\frac{\partial T}{\partial q_i} = (\frac{1}{2} m)\frac{\partial}{\partial q_j}(\dot{r}\cdot\dot{r}]$

$\partial T/\partial q_j =$ [illegible]

(ii) Kirchhoff's Laws.
(iii) Phase transitition
(iv)✓ Gibb's paradox

VII. (a) Show that the relation between the scattering angles as observed in the Center of mass Coordinate System & laboratory coordinate system i.e. Q_c & Q_L when the two interacting masses m_1 & m_2 are nearly equal is: $\tan\theta_1 = \tan\theta/2$ (05)

(b) Two Particles having masses m_1 & m_2 move so that their relative Velocity is V & Velocity of their Centre of mass is V'. If $M = m_1 + m_2$ & the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

Then Prove that their $K.E = \frac{1}{2}MV'^2 + \frac{1}{2}\mu V^2$ (05)

VIII. Define the three Kepler laws & derive the defining equation of any one of them. (10)

IX. Show that for the Poisson Brackets: (10)

(a) $[U,U]=0$, (b) $[U,C]=0$ C = Constant (c) $[U,V_1+V_2]=[U,V_1]+[U,V_2]$

(d) $[U,VW]=[U,V]W+V[U,W]$ (e) $\frac{\partial}{\partial t}[U,V]=\left[\frac{\partial u}{\partial t},V\right]+\left[U,\frac{\partial v}{\partial t}\right]$ (10)

X. Write short note on any Two of the following: (05+05)

(a) D, Alembert's Principle.
(b) Cyclic Coordinates.
(c) [illegible]

فون: 091-5703445
عوامی فوٹوسٹیٹ
میٹرک، ایف اے، ایف ایس سی، بی اے، بی ایس سی، ایم اے، ایم ایس سی، کے نوٹس دستیاب ہیں
جان 0321-9103972
0345-9223209
0300-5893473

UNIVERSITY OF PESHAWAR
M.Sc. PHYSICS (PREVIOUS)
ANNUAL EXAMINATION (2006)
PAPER – III: CLASSICAL MECHANICS

Time Allowed: 04 Hours
Maximum Marks: 60

Note: Attempt any Five questions. All questions carry equal marks.

Possession & use of Mobiles & other Electronic accessories is strictly prohibited during the Examination & can lead to Cancellation of Paper as well as Examination on Supdt's report.

I. (a) For a System of Two Bodies, show that center of mass is either at rest or moves with uniform speed i.e. $M\ddot{r}_c = 0$ (05)

(b) Prove that the Time derivative of the Total Angular Momentom of the System is equal to the external Torque. (05)

II. (a) Define the Following Terms:
(i) Constraints
(ii) Dynamical System
(iii) Generalized Coordinates
(iv) Conservative Force
(v) Degrees of Freedom
(05)

(b) A particle of Constt: mass moves under the influence of a Force Field, F, Assuming at time t_1 & t_2, the velocity is V_1 & V_2 respectively. Prove that the work done on the particle results in the change in its Kinetic Energy. (05)

III. (a) Derive the Lagranges equation for the mechanical system with "n" degrees of freedom acted upon by the Conservative Force. (07)

(b) Set up a Lagranges Equation for one dimensional 1-D Harmonic Osciallator acted upon by a Force, $F = -Kx$. (03)

IV. (a) Derive Lagranges equation from Hamilton's Principle. (05)

(b) Set up a Lagrangian for a simple pendulum of length "ℓ" & obtain equation of motion for the system. (05)

V. (a) If the Hamiltonian "H" $= \sum_{i=1}^{n} p_i \dot{q}_j - L$ is expressed as function of coordinates of q & momenta p_j, then Prove that $\dot{p}_j = \frac{-\partial H}{\partial q_j}$ $\dot{q}_j = \frac{\partial H}{\partial p_j}$ (06)

(b) A Particle located by (r, θ), moving in the (x, y) plane moving under the influence of a central force (potential $V(r)$). Setup the Hamiltonian & obtain the Hamilton's equations of motion. (04)

VI. Using Variational Principle show that the shortest distance between two points is a straight line. (10)

(P.T.O)

فون:091-5703445
عوامی فوٹوسٹیٹ
میٹرک، ایف اے، ایف ایس سی، بی اے، بی ایس سی، ایم اے، ایم ایس سی، کے نوٹس دستیاب ہیں۔
پروپرائٹر:
0321-9103972 جان
0345-9223209
0300-5893473

---

(44)

$P = kd$
$P = 17 \times 7$
$P = 119$

∴ When Pressure = 119 N/cm² then
d = 7cm.

Q3 Pressure P of a gas in a container varies directly as temperature. T When Pressure is 50 N/cm², temperature is 75°C. Find Pressure when temperature is 150°C.

Solution

As Pressure = P
Temperature = T

As there is direct variation between Pressure and temperature Mathematically

$P \propto T$
$P = kT \rightarrow (1)$

To find k. Put $P = 50\ N/m^2$, $T = 75°C$
in eq (1)
$P = kT$
$50 = k \times 75$
Divide both sides by 75